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Algebraic Properties of Endomorphisms of Abelian Groups and Rings

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ALGEBRAIC PROPERTIES OF ENDOMORPHISMS OF ABELIAN GROUPS AND RINGS

by

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Johnnie G. Slagle

NOTATION AND DEFINITIONS OF TERMS

USED IN CHAPTERS I-III

(0.1) Definitions. If X and Y are sets then $X \times Y = \{(x,y) \mid x \in X,\ y \in Y\}$. A relation is any set of ordered pairs. A relation f is single-valued iff if $(x,y),(x,z) \in f$ then $y=z$. A single-valued relation is a function. The set $f^{-1} = \{(x,y) \mid (y,x) \in f\}$ where f is a relation is the inverse of f. If f and f^{-1} are both functions then f is one-to-one (1-1). The set $\mathrm{dom} f = \{x \mid (x,y) \in f \text{ for some } y\}$ is the domain of f, and $\mathrm{rng} f = \{y \mid (x,y) \in f \text{ for some } x\}$ is the range of f. If $\mathrm{dom} f = X$, $\mathrm{rng} f \subset Y$, and f is a function then f is said to be a function on X into Y and we write $f: X \to Y$. f is a function on X iff $f \subset X \times Y$, f is a function, and $\mathrm{dom} f = X$. f is a function on X onto Y iff $f \subset X \times Y$ is a function on X and $\mathrm{rng} f = Y$. The image of A under f is the set $f(A) = \{y \mid (x,y) \in f \text{ for some } x \in A\}$.

(0.2) Notation. Let wrt be an abbreviation of with respect to, and iff of if and only if. $A \subsetneq B$ means A is a proper subset of B. N will denote the natural numbers and I the integers. $A-B = \{x \in A \mid x \in B\}$. If $(R,+,\cdot)$ and $(\bar{R},\bar{+},\bar{\cdot})$ are rings then $+$ and $\bar{+}$ will both be written as $+$ and $\cdot$ and $\bar{\cdot}$ as $.$ or no symbol. When no confusion will arise R will be written for $(R,+,\cdot)$. If $(R,+,.)$ is a set with two binary operations on it then $na = a + \ldots + a$ n-times and $a^n = a \cdot \ldots \cdot a$ n-times.

(0.3) Definitions. If M is a set then # is a binary operation on M iff (a) $\# \subset (MxM)xM$ and (b) # is a function on MxM into M. If M is a set with binary operations $\#_i$, $i=1,2,\ldots,n$, on it, N is a set with binary operations o_i, $i=1,2,\ldots,n$, on it, and $\alpha \subset MxN$ is a function on M into N then α is a homomorphism of M into N iff $(a\#_i b)\alpha = (a\alpha)o_i(b\alpha)$ for $i=1,2,\ldots,n$; and α is an isomorphism iff α is a homomorphism that is one-to-one. If M=N, $\#_i = o_i$ for $i=1,2,\ldots,n$ then α is an endomorphism of M iff α is a homomorphism of M into N; and α is an automorphism iff α is an endomorphism that is one-to-one and onto. If α is a homomorphism of a group or a ring into a group or ring respectively then the kernal of α is the set $K_\alpha = x \quad x\alpha = o$, where o is the additive identity .

(0.4) Notation. Let $E(R,\#_1,\ldots,\#_n)$ be the set of endomorphisms of $(R,\#_1,\ldots,\#_n)$, and when no ambiguity will arise let E(R) denote the set. Also, let $A(R,\#_1,\ldots,\#_n)$ be the set of automorphisms.

(0.5) Definitions. If M is a set and (G,#) is a group then M-(G,#) is an M-Group wrt η iff η (GxM)xG is a function on GxM such that for $x,y \in G$ and $m \in M$ $(x\#y,m)\eta = (x,m)\eta\#(y,m)\eta$. If $(R,+,\cdot)$ and $(\overline{R},\overline{+},\overline{\cdot})$ are rings then R is embedded in $\overline{R}$ iff R is isomorphic to a subring of $\overline{R}$. If (R,+) is a group and $S \subset R$ then $x+S = \{x+s \mid s \in S\}$ and $R/S = \{x+S \mid x \in R\}$.

ABSTRACT

Algebraic Properties of Endomorphisms
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The main objective of the thesis was to extend the definition of an M-Group to what is called an M-Ring. From this extension a system called an expanded ring follows naturally. To facilitate the development of the expanded ring, chapter I is devoted to developing properties on systems that are not quite rings where many interesting examples are constructed. In chapter II the definition of an M-Ring is given and some of its properties are derived. In chapter III some of the properties of expanded rings are proved, and examples of expanded rings are given to show their existence.

(81 pages)

INTRODUCTION

The main objective of this thesis is to extend the definition of an M-Group to what is called here an M-Ring. From this extension a system called an expanded ring follows naturally. To facilitate the development of the expanded ring, chapter I is devoted to developing properties on systems that are not quite rings where many interesting examples are constructed. In chapter II the definition of an M-Ring is given and some of its properties are derived. In chapter III some of the properties of expanded rings are proved, and examples of expanded rings are given to show their existence.

CHAPTER I

SEMI-RINGS AND M-GROUPS

The definitions of bare rings and semi-rings are given, along with a theorem concerning their equivalence with a class of M-Groups. The examples are listed in the last section of the chapter, and structural properties are discussed by referring to the proper examples.

Section 1. Right semi-rings

(1.1) Definitions. $(R,+,\cdot)$ is a bare ring (shortly BR) iff $(R,+)$ is an abelian group and $\cdot$ is a binary operation on R. If $(R,+,\cdot)$ is a BR then $(R,+,\cdot)$ is a right semi-ring (RSR) iff for any $x,y,z \in R$ $(x+y)\cdot z=x\cdot z+y\cdot z$. $(R,+,\cdot)$ is a commutative RSR iff $(R,+,\cdot)$ is a RSR that is commutative wrt $\cdot$.

(1.2) Remark. From the definition, an M-Group is dependent upon the map η. Hence, many M-Groups may be formed from a set M and a group G. Also, from Jacobson (1951) an M-Group can be equivalently defined as the following. If M is a set and $(G,+)$ is a group then $M-(G,+)$ is an M-Group wrt η iff there exists a function α on M into $E(G)$ such that $x(m\alpha)=(x,m)\eta$ for $x \in G$ and $m \in M$. So as to make the dependency of an M-Group, $M-(G,+)$, upon its mappings be more explicit let $M_\eta^\alpha-(G,+)$ denote the M-Group. When no confusion can arise we sometimes will denote it by M-(G,+) or M-G.

(1.3) Theorem. If $(R,+,\circ)$ is a RSR and $\eta=\circ$ then R_{η}^{α}-(R,+) is an M-Group.

Proof. η is a function on RxR since $\circ$ is a binary operation on R and for $x,y,z \in R$ $(x+y,z)\eta=(x+y)\circ z= x\circ z+y\circ z=(x,z)\eta+(y,z)\eta$.

(1.4) Theorem. If R_{η}^{α}-(R,+) is an M-Group and $\circ=\eta$ then $(R,+,\circ)$ is a RSR.

Proof. Since $\eta \subset (R\mathrm{x}R)\mathrm{x}R$ is a function on RxR $\circ$ is a binary operation on R. Furthermore, for $x,y,z \in R$ $(x+y)\circ z=(x+y,z)\eta=(x,z)\eta+(y,z)\eta=x\circ z+y\circ z$.

In view of the theorems (1.3) and (1.4) we can consider a RSR as a special M-Group and any M-Group, M-G, with M=G as a RSR.

(1.5) Notation. For an M-Group, M-G, denote $M\alpha$ as $\overline{M} \subset E(G)$ and $m\alpha$ for $m \in M$ as $\overline{m}$.

(1.6) Theorem. If M_{η}^{α}-(R,+) is an M-Group then

(a) $(o,m)\eta=o(m\alpha)=o$ for $m \in M$,

(b) $(kx,m)\eta=(kx)(m\alpha)=k[x(m\alpha)]=k(x,m)\eta$ for $x \in R$, $m \in M$ and $k \in I$,

(c) $(-x,m)\eta=(-x)(m\alpha)=-[x(m\alpha)]=-(x,m)\eta$ for $x \in R$ and $m \in M$.

Proof. Since $m\alpha=\overline{m}$ is an endomorphism the results are obvious.

(1.7) Definitions. If $(R,+,\circ)$ is a BR then r is a right (left) identity of R wrt $\circ$ iff for any $x \in R$ $x\circ r=x$ $(r\circ x=x)$. R has a two sided identity r wrt $\circ$ iff r is both a left and a right identity of R wrt $\cdot$.

(1.8) Theorem. If R is a RSR with a left identity r and $R \neq \{o\}$ then $r \neq o$.

Proof. Assume $r=o$. Since $R \neq \{o\}$ there exists an $x \in R$ such that $x \neq o$. From theorem (1.6) $x=r \cdot x=o \cdot x=o$.

(1.9) Remark. If R is a RSR with a right identity then r need not be different from zero. Such is the case in example (6.2).

(1.10) Theorem. If R is a BR, r is a right identity, and e is a left identity then $r=e$.

Proof. $e=e \cdot r=r$.

(1.11) Theorem. If R is a BR and e is a two sided identity then e is unique and any right (left) identity of R is equal to e.

Proof. Assume there exists $e' \in R$ such that $e' \neq e$ and e' is a two sided identity. Then $e=e \cdot e'=e'$. Let r be a right identity of R. Then $e=e \cdot r=r$.

(1.12) Remarks. From the above theorems a BR that has more than one right (left) identity cannot have a left (right) identity.

Since a RSR is equivalent to a special M-Group, namely $R^{\alpha}_{\eta}-(R,+)$, the notion of identities is equivalent to the following. r is a right identity of R iff $r \in H_{\alpha}=\{x \in R \mid l(x\alpha)=I \in E(R)$ where $xI=x$ for all $x \in R\}$, and e is a left identity of R iff for every $\bar{x} \in R\alpha \subset E(R)$ $e\bar{x}=x$. If $H_{\alpha}=\emptyset$ then R has no right identity (look at examples (6.1) and (6.3)). From above it appears that the condition on the left identity may be so strong that a left identity may not exist for any RSR, but example (6.3) shows that this assertation is false. Furthermore, one might ask whether there exists a RSR that has a left identity but no right. Such is the case in example (6.3).

(1.13) Theorem. If R is a RSR with R_{η}^{α}-R as its M-Group equivalent and R has a left identity e then (a) α is one-to-one and (b) if R has no right identity then $I \notin R\alpha = \overline{R}$ where for $x \in R$ $xI = x$.

Proof. (a) Let $\overline{x}, \overline{y} \in \overline{R}$ such that $\overline{x} = \overline{y}$. Then $x = ex = e\overline{x} = e\overline{y} = ey = y$. Hence, $x\alpha = y\alpha$ implies $x = y$. (b) Assume $I \in \overline{R}$. Then there exists $x \in R$ such that $x\alpha = I = \overline{x}$. Which implies $yx = y\overline{x} = yI = y$ for all $y \in R$. Thus, x is a right identity, but R contains no right identities.

(1.14) Definitions. If R is a BR then (a) $x \in R$ is a right divisor of zero (zero divisor) iff there exists $y \in R$ such that $y \neq o$ and $yx = o$, and (b) $x \in R$ is a left divisor or zero iff there exists $y \in R$ such that $y \neq o$ and $xy = o$.

(1.15) Remarks. In a RSR zero is always a left divisor of zero by theorem (1.6), but as shown in example (6.1) zero is not always a right divisor of zero.

Again in terms of an M-Group we have that $x \in R$ is a right divisor of zero iff $K_{\overline{x}} = \{y \in R \mid y\overline{x} = o\} \supsetneq \{o\}$ and $x \in R$ is a left divisor of zero iff there exists a $\overline{y} \in \overline{R} \subset E(R)$ such that $x\overline{y} = o$ (i.e. $x \in K_{\overline{y}}$) and $y \neq o$.

If $x \in R$ is a left divisor of zero then x need not be a right divisor of zero as shown by example (6.2) and if $x \in R$ is a right divisor of zero then x need not be a left divisor of zero as shown by example (6.3).

(1.16) Theorem. If R is a RSR, $R \neq \{o\}$, and R has a right identity r then r is not a right divisor of zero.

Proof. Let $x \in R$ such that $x \neq o$. Then $x \circ r = x \neq o$.

(1.17) Remark. If R is a RSR, $R \neq \{o\}$, and R has a right identity r then as shown by example (6.2) it may be a left divisor of zero.

(1.18) Definitions. If R is a BR then $x \in R$ obeys the right cancellation law iff when $y, z \in R$ such that $yx = zx$ then $y = z$; $x \in R$ obeys the left cancellation law iff when $y, z \in R$ such that $xy = xz$ then $y = z$.

(1.19) Remark. In example (6.2) for $x = -1$ we see that if x obeys the left cancellation law it need not obey the right cancellation law, and also in example (6.2) for $x = o$ we see that if x obeys the right cancellation law it need not obey the left cancellation law.

Zero does not obey the left cancellation law unless $R = \{o\}$. However, as shown in example (6.2) o may obey the right cancellation law even when $R \neq \{o\}$.

(1.20) Theorem. If R is a RSR then $x \in R$ obeys the right cancellation law iff x is not a right divisor of zero.

Proof. If $yx = o$ for some $y \in R$ then $yx = ox$ and hence $y = o$ when the cancellation law holds. Also, if $yx = zx$ for some $y, z \in R$ then $(y-z)x = o$ and hence $y = z$ when x is not a right divisor of zero.

(1.21) Remarks. From theorem (1.20) if we know what the set, D_r, of right diviors of zero is, then the set of elements, C_r, that obeys the right cancellation law is R-D_r, and vice versa.

A natural question is whether a similar theorem to (1.20) exists when right is replaced by left. In example (6.2) every element $x \neq o$ obeys the left cancellation law, but also is a left divisor of zero. In example (6.3) $x=(m,-m)$ $m \neq o$ is not a left divisor of zero, and furthermore x does not obey the left cancellation law. Hence, such a theorem does not exist. Furthermore, as shown no theorem with the same hypothesis as theorem (1.20) exists when either right is replaced by left. If x obeys the right cancellation law then x may be a left divisor of zero or not as shown by example (6.2) for $x \neq -1$ and by example (6.1) for $x=(a,b)$ where $a,b \neq o$. If x is not a right divisor of zero then it need not obey the left cancellation law as shown by example (6.3) for $x=(m,-m)$ $m \neq o$. However, if x is not a right divisor of zero then x can obey the left cancellation law as shown by example (6.2) for $x \neq -1,o$.

(1.22) Definitions. If R is a RSR and R has a right (left) identity r wrt $\cdot$ then y is a right inverse of x wrt $\cdot$ relative to r iff $xy=r$, and y is a left inverse of x wrt $\cdot$ relative to r iff $yx=r$.

(1.23) Remark. Considering R as an M-Group then y is a right inverse of x iff $\bar{y}$ maps x onto r, and y is a left inverse of x iff $\bar{x}$ maps y onto r.

Section 2. Semi-rings and n-multiple M-Groups

(2.1) Definitions. If $(R,+,\cdot)$ is a BR then $(R,+,\cdot)$ is a left semi-ring (LSR) iff for any $x,y,z \in R$ $z(x+y)=zx+zy$.

$(R,+,\cdot)$ is a semi-ring (SR) iff R is a RSR and a LSR.

(2.2) Remark. A LSR is really the same as a RSR except for notational changes. Hence, all of the properties of a RSR carries over for a LSR.

(2.3) Definition. If $(G,+)$ is a group then $M^{\alpha_1,\ldots,\alpha_n}_{\eta_1,\ldots,\eta_n}$-$(G,+)$ is a n-multiple M-Group wrt $\eta_1,\ldots,\eta_n$ and $\alpha_1,\ldots,\alpha_n$ iff $M^{\alpha_i}_{\eta_i}$-$(G,+)$ is an M-Group for $i=1,\ldots,n$. If $G^{\alpha_1,\alpha_2}_{\eta_1,\eta_2}$-$(G,+)$ is a 2-multiple M-Group and $\sqrt{\ } \subset (GxG)x(GxG)$ is a function on GxG such that $(x,y)\sqrt{\ }=(y,x)$ for $x,y \in G$ then $G^{\alpha_1,\alpha_2}_{\eta_1,\eta_2}$-$(G,+)$ is a symmetric 2-multiple M-Group iff $\eta_1=\sqrt{\ }\eta_2$.

(2.4) Theorem. If $(R,+,\cdot)$ is a SR and $\eta_1=\cdot$ then $R^{\alpha_1,\alpha_2}_{\eta_1,\eta_2}$-$(R,+)$ where $\eta_2=\sqrt{\ }\eta_1$ is a symmetric 2-multiple M-Group.

Proof. Clearly, η_1 and η_2 are both functions on RxR into R. For $x,y,z \in R$ $(x+y,z)\eta_1=(x+y)z=xz+yz=(x,z)\eta_1+(y,z)\eta_1$ and $(x+y,z)\eta_2=(z,x+y)\eta_1=z(x+y)=zx+zy=(z,x)\eta_1+(z,y)\eta_1=(x,z)\eta_2+(y,z)\eta_2$.

(2.5) Theorem. If $R^{\alpha_1,\alpha_2}_{\eta_1,\eta_2}$-$(R,+)$ is a symmetric 2-multiple M-Group and $\cdot=\eta_1$ then $(R,+,\cdot)$ is a SR.

Proof. Clearly $\cdot$ is a binary operation on R. For $x,y,z \in R$ $(x+y)z=(x+y,z)\eta_1=(x,z)\eta_1+(y,z)\eta_1=xz+yz$ and $z(x+y)=(z,x+y)\eta_1=(x+y,z)\eta_2=(x,z)\eta_2+(y,z)\eta_2=(z,x)\eta_1+(z,y)\eta_1$ $=zx+zy$.

From theorems (2.4) and (2.5) we can consider a SR as a symmetric 2-multiple M-Group and vice versa.

(2.6) Theorem. If $\bar{+} \subset (E(R,+)xE(R,+))xF$, $F \subset RxR$, and $(R,+)$ is an abelian group such that for $\alpha,\beta \in E(R)$ and $x \in R$ $x((\alpha,\beta)\bar{+})=x\alpha+x\beta$ then $(E(R,+),\bar{+})$ is an abelian group.

Proof. Let $(\alpha,\beta),(\alpha',\beta') \in E(R)xE(R)$ such that $(\alpha,\beta)=(\alpha',\beta')$. Then $\alpha=\alpha'$ and $\beta=\beta'$, and hence for arbitrary $x \in R$ $x((\alpha,\beta)\bar{+})=x\alpha+x\beta=x\alpha'+x\beta'=x((\alpha',\beta')\bar{+})$ which implies that $(\alpha,\beta)\bar{+}=(\alpha',\beta')\bar{+}$. Therefore $\bar{+}$ is a function. For $x,y \in R$ $(x+y)((\alpha,\beta)\bar{+})=(x+y)\alpha+(x+y)\beta=(x\alpha+y\alpha)+(x\beta+y\beta)=(x\alpha+x\beta)+(y\alpha+y\beta)$ $=[x(\alpha,\beta)\bar{+}]+[y(\alpha,\beta)\bar{+}]$. Therefore $(\alpha,\beta)\bar{+} \in E(R)$ and hence $F \subset E(R)$.

Clearly since $(R,+)$ is commutative and associative $(E(R),+)$ is too. The function $\bar{o} \subset RxR$ on R such that for $x \in R$ $x\bar{o}=o$ is also in $E(R)$, and for $x \in R$ and $\alpha \in E(R)$ $x(\bar{o}\bar{+}\alpha)=x\bar{o}+x\alpha=x\alpha$. Therefore $\bar{o}$ is the identity for $E(R)$. Clearly for $\alpha \in E(R)$ $(-\alpha) \subset RxR$ such that for $x \in R$ $x(-\alpha)=-(x\alpha)$ is in $E(R)$. Thus for $x \in R$ and $\alpha \in E(R)$ $x[\alpha\bar{+}(-\alpha)]=x\alpha+x(-\alpha)=o=x\bar{o}$ which implies $\alpha\bar{+}(-\alpha)=\bar{o}$. Hence, $(E(R),\bar{+})$ is an abelian group.

(2.7) Theorem. If $(R,+,\cdot)$ is a SR with $R^{\alpha_1,\alpha_2}_{\eta_1,\eta_2}-(R,+)$ as its symmetric 2-multiple M-Group then α_1 (α_2) is a homomorphism of $(R,+)$ into $(E(R),\bar{+})$.

Proof. We know α_1 is a function of $(R,+)$ into $E(R)$. For $x,y,z \in R$ $z((x+y)\alpha_1)=(z,x+y)\eta_1=(x+y,z)\eta_2=(x,z)\eta_2+(y,z)\eta_2=(z,x)\eta_1+(z,y)\eta_1=z(x\alpha_1)+z(y\alpha_1)=z[(x\alpha_1)\bar{+}(y\alpha_1)]$. Since z was arbitrary $(x+y)\alpha_1=(x\alpha_1)\bar{+}(y\alpha_1)$. Hence, α_1 is a homomorphism and hence $R\alpha_1=\bar{R} \subset E(R)$ is a subgroup of $E(R)$.

(2.8) Remarks. From theorem (2.7) every abelian group (R,+) that has a homomorphism α of itself onto a subgroup of (E(R),+) can be considered a SR if $\cdot$ is defined as $x \cdot y = x(y\alpha) = x\bar{y}$ for all $x,y \in R$.

Section 3. Right rings (RR)

(3.1) Definitions. If $(R,+,\cdot)$ is a RSR then R is a right ring (RR) iff the operation $\cdot$ is associative. If R is a SR then R is a ring iff the operation $\cdot$ is associative.

(3.2) Remark. Most of the theorems about RSRs hold for RRs and similarly for SRs and rings. The additional property of associativity will permit the proof of more theorems on inverses and divisors of zero, but only one will be given.

(3.3) Theorem. If R is a RR with a right identity r, and x has a right inverse then x is not a right divisor of zero.

Proof. Let y be the right inverse of x. Assume x is a right divisor of zero. Then there exists a $z \in R$ such that $z \neq o$ and $zx = o$. Hence, $xy = r$ implies $z = zr = z(xy) = (zx)y = oy = o$.

Section 4. Ideals and homomorphisms

(4.1) Notation. When BR is used in the same sentence several times it refers to the same type of BR under discussion.

(4.2) Definitions. If $(R,+,\cdot)$ is a BR and $\emptyset \neq S \subset R$ then $(S,+,\cdot)$ is a subBR iff $(S,+,\cdot)$ is a BR. $S \neq \emptyset$ is a right (left) ideal of the BR R iff (S,+) is a subgroup of (R,+) and $xs \in S$ $(sx \in S)$ for all $x \in R$ and for all $s \in S$.

S is an ideal of the BR R iff S is both a right ideal of R and a left ideal of R.

(4.3) Theorem. If $(R,+,\circ)$ is a BR and $\emptyset \neq S \subset R$ then S is a subBR iff for $x,y \in S$ $x-y \in S$ and $xy \in S$.

Proof. Obvious.

(4.4) Theorem. If $(R,+,\circ)$ is a SR, $\emptyset \neq S \subset R$ is an ideal of R and $\oplus, \odot \subset (R/S \times R/S) \times R/S$ such that for $x=a+S$, $y=b+S \in R/S$ $(x,y)\oplus=(a+b)+S$ and $(x,y)\odot=ab+S$ then $(R/S,\oplus,\odot)$ is a SR and $\odot$ is associative (commutative) iff $\circ$ is associative (commutative).

Proof. First we need to prove that $\oplus$ and $\odot$ are binary operations. If $\alpha=(a+S,b+S), \beta=(a'+S,b'+S) \in R/S \times R/S$ such that $\alpha=\beta$ then $a+S=a'+S$ and $b+S=b'+S$. Hence there exists $s,t \in S$ such that $a=a'+s$ and $b=b'+t$. Because R is a SR and S is an ideal of R $a+b=(a'+b')+(s+t)$ and $ab=(a'+s)(b'+t)=a'b'+(a't+sb'+st)$ which implies that $(a+b)+S=(a'+b')+S$ and $ab+S=a'b'+S$. Hence, $\alpha\oplus=(a+b)+S=(a'+b')+S=\beta\oplus$ and $\alpha\odot=ab+S=a'b'+S=\beta\odot$.

Clearly $(R/S,\oplus)$ is an abelian group since $(R,+)$ is an abelian group. For $x=a+S, y=b+S, z=c+S \in R/S$

(a) $(x\oplus y)\odot z=[(a+b)+S]\odot z=(a+b)c+S=(ac+bc)+S=(ac+S)\oplus(bc+S)$
$=(x\odot z)\oplus(y\odot z)$

(b) $z\odot(x\oplus y)=z\odot[(a+b)+S]=c(a+b)+S=(ca+cb)+S=(ca+S)\oplus(cb+S)$
$=(z\odot x)\oplus(z\odot y)$.

Hence, $(R/S,\oplus,\odot)$ is a SR. Also from

(c) $(x\odot y)\odot z=(ab+S)\odot z=(ab)c+S$
$x\odot(y\odot z)=x\odot(bc+S)=a(bc)+S$

(d) $x\odot y=ab+S$ and $y\odot x=bc+S$

we see that $\circledcirc$ is associative (commutative) iff $\cdot$ is associative (commutative).

(4.5) Remark. If R had been a RSR in theorem (4.5) instead of a SR then $(R/S, \oplus, \circledcirc)$ need not be a RSR since $\circledcirc$ is not necessarly a binary operation on R/S.

$(R/S, \oplus, \circledcirc)$ is called the factor SR of the SR $(R,+,\cdot)$ wrt the ideal S.

(4.6) Theorem. If η is a homomorphism of the BR $(R,+,\cdot)$ into the BR $(\overline{R},\overline{+},\overline{\cdot})$ then $R\eta \subset \overline{R}$ is a subBR.

Proof. Since η is a homomorphism we know that $(R\eta, \overline{+})$ is an abelian subgroup of $(\overline{R},\overline{+})$. For $x=a\eta, y=b\eta \in R\eta$ $x\overline{\cdot}y = (a\eta)\overline{\cdot}(b\eta)=(ab)\eta \in R\eta$. Hence, by theorem (4.4) $(R\eta,\overline{+},\overline{\cdot})$ is a BR.

(4.7) Theorem. If η is a homomorphism of the RSR (LSR) $(R,+,\cdot)$ into the RSR (LSR) $(\overline{R},\overline{+},\overline{\cdot})$ then the kernal, K_η, of η is a left (right) ideal of the RSR (LSR) R.

Proof. We know that $(K_\eta,+)$ is a subgroup of $(R,+)$. Let $x \in R$ and $a \in K_\eta$. Then $(ax)\eta=(a\eta)\overline{\cdot}(x\eta)=\overline{o}\,\overline{\cdot}(x\eta)=\overline{o}$. Therefore, $ax \in K_\eta$ and hence K_η is a left ideal of R.

(4.8) Theorem. If η is a homomorphism of the SR $(R,+,\cdot)$ into the SR $(\overline{R},\overline{+},\overline{\cdot})$ then K_η is an ideal of R.

Proof. Follows from theorem (4.7).

(4.9) Theorem. If $(R,+,\cdot)$, $(\overline{R},\overline{+},\overline{\cdot})$ and $(\overline{\overline{R}},\overline{\overline{+}},\overline{\overline{\cdot}})$ are BRs, and $\pi \subset R x \overline{R}$ and $\overline{\pi} \subset \overline{R} x \overline{\overline{R}}$ are homomorphisms on R and on $\overline{R}$ respectively then $\pi\overline{\pi}$ (the resultant) is a homomorphism on R into $\overline{\overline{R}}$.

Proof. If $x,y \in R$ then $(x+y)(\pi\overline{\pi})=((x+y)\pi)\overline{\pi}=(x\pi+y\pi)\overline{\pi}$ $=(x\pi)\overline{\pi}+(y\pi)\overline{\pi}=x(\pi\overline{\pi})+y(\pi\overline{\pi})$ and $(xy)(\pi\overline{\pi})=((xy)\pi)\overline{\pi}=[(x\pi)(y\pi)]\overline{\pi}$

$=[(x\pi)\bar{\pi}][(y\pi)\bar{\pi}]=[x(\pi\bar{\pi})][y(\pi\bar{\pi})]$. Therefore, $\pi\bar{\pi}$ is a homomorphism.

(4.10) Theorem. If $(\bar{R},\oplus,\circledast)$ is the factor SR of the SR $(R,+,\circ)$ wrt the ideal S and $\tau\subset R\times R$ such that for $x\in R$ $x\tau=x+S$ then τ is a homomorphism.

Proof. If $x,y\in R$ such that $x=y$ then $x\tau=x+S=y+S=y\tau$. Therefore, τ is a function on R. Again if $x,y\in R$ then $(x+y)\tau=(x+y)+S=(x+S)\oplus(y+S)=(x\tau)\oplus(y\tau)$ and $(xy)\tau=xy+S=(x+S)\circledast(y+S)=(x\tau)\circledast(y\tau)$. Hence, τ is a homomorphism.

(4.11) Remark. τ as defined in theorem (4.10) is called the natural homomorphism on R into $\bar{R}=R/S$.

(4.12) Theorem. If R and R' are SRs, $\varphi\subset R\times R'$ is a homomorphism on R, $\bar{R}$ is the factor SR of R wrt the ideal $S\subset K_\varphi$, and $\bar{\varphi}\subset\bar{R}\times R'$ such that $(x+S)\bar{\varphi}=x\varphi$ for all $x\in R$ then (a) $\bar{\varphi}$ is a homomorphism on $\bar{R}=R/S$ into R' and $\varphi=\tau\bar{\varphi}$ where τ is the natural homomorphism on R into $\bar{R}$, and (b) $\bar{\varphi}$ is a isomorphism iff $K_\varphi=S$.

Proof. If $x+S,y+S\in\bar{R}$ such that $x+S=y+S$ then $x=y+s$ for some $s\in S$, and $(x+S)\bar{\varphi}=x\varphi=(y+S)\varphi=y\varphi+s\varphi=y\varphi=(y+S)\bar{\varphi}$. Hence, $\bar{\varphi}$ is a function on $\bar{R}$. Also, if $x+S,y+S\in\bar{R}$ then $[(x+S)\oplus(y+S)]\bar{\varphi}=[(x+y)+S]\bar{\varphi}=(x+y)\varphi=x\varphi+y\varphi=(x+S)\bar{\varphi}+(y+S)\bar{\varphi}$ and $[(x+S)\circledast(y+S)]\bar{\varphi}=(xy+S)\bar{\varphi}=(xy)\varphi=(x\varphi)(y\varphi)=(x+S)\bar{\varphi}(y+S)\bar{\varphi}$. Therefore, $\bar{\varphi}$ is a homomorphism on $\bar{R}$. The natural homomorphism on R into $\bar{R}$ is such that $x\tau=x+S$ for $x\in R$. $\tau\bar{\varphi}\subset R\times R'$ is a homomorphism on R by theorem (4.9. Let $x\in R$ then $x(\tau\bar{\varphi})=(x\tau)\bar{\varphi}=(x+S)\bar{\varphi}=x\varphi$. Since x was arbitrary $\tau\bar{\varphi}=\varphi$.

If $\bar{\varphi}$ is an isomorphism then it is one-to-one. Let

$x \in K_\varphi$. Then $o=x\varphi=(x+S)\bar{\varphi}=o\varphi=(o+S)\bar{\varphi}$. Which implies $x+S=o+S$ or i.e. $x \in S$. Therefore, $K_\varphi \subset S$. By hypothesis $S \subset K_\varphi$. Hence, $S=K_\varphi$. Let $K_\varphi=S$. For $\bar{x}=x+S, \bar{y}=y+S \in \bar{R}$ such that $\bar{x}\bar{\varphi}=\bar{y}\bar{\varphi}$ then $x\varphi=y\varphi$ which implies $(x-y)\varphi=o$ or i.e. $x-y \in K_\varphi=S$. Thus, $\bar{x}=x+S=y+S=\bar{y}$. Therefore, $\bar{\varphi}$ is one-to-one and hence an isomorphism.

(4.13) Remark. In theorem (4.12) if $R'=R\varphi$ and $S=K_\varphi$ then $\bar{\varphi}$ is an isomorphism of $\bar{R}$ onto R'.

Section 5. An embedding theorem

The purpose of this section is to derive an embedding theorem that will help us in finding examples of rings with identities .

(5.1) Theorem. If $(R,+,\circ)$ is a BR, $A=IxR$, and $+,\circ$ are subsets of $(AxA)xA$ such that for $(m,a),(n,b) \in A$

(a) $((m,a),(n,b))+=(m+n,a+b)$

and (b) $((m,a),(n,b))\circ=(mn,na+mb+ab)$,

where $na=a+\ldots+a$ n-times, then + and $\circ$ are binary operations on A and $(A,+)$ is an abelian group.

Proof. Let $x=((m,a),(n,b)), y=((m',a'),(n',b')) \in AxA$ such that $x=y$. Now $x=y$ implies $m'=m, n'=n, a'=a$, and $b'=b$. Thus $x+=(m+n,a+b)=(m'+n',a'+b')=y+$ and $x\circ=(mn,na+mb+ab)=(m'n',n'a'+m'b'+a'b')=y\circ$. Therefore, + and $\circ$ are binary operations on A.

The proof that $(A,+)$ is and abelian group is obvious.

(5.2) Theorem. If $(A,+)$ is the abelian group in theorem (5.1) and $\circ$ is the binary operation defined on A in theorem (5.1) then

(a) A is commutative wrt $\circ$ iff R is commutatuve wrt $\circ$,

(b) every element of A obeys the right (left) distributive law iff every element of R obeys the right (left) distributive law,

(c) (1,o) is a right (left) identity iff xo=o (ox=o) for all $x \in R$,

and (d) A is associative only if R obeys the distributive laws and is associative.

Proof. Let (m,a),(n,b),(k,c) be arbitrary elements in A.

(a) Since (m,a)(n,b)=(mn,na+mb+ab) and (n,b)(m,a)=(nm,mb+na+ba)=(mn,na+mb+ba) it is clear that A is commutative iff R is commutative.

(b) Since [(m,a)+(n,b)](k,c)=[(m+n)k,k(a+b)+(m+n)c+(a+b)c]=[mk+nk,ka+kb+mc+nc+(a+b)c] and (m,a)(k,c)+(n,b)(k,c)=(mk,ka+mc+ac)+(nk,kb+nc+bc)=(mk+nk,ka+kb+mc+nc+ac+bc) it is clear that A obeys the right distributive law iff R obeys the right distributive law.

(c) Now because (m,a)(1,o)=(m,a+ao) and (1,o)(m,a)=(m,a+oa) we see that (1,o) is a right (left) identity iff xo=o (ox=o) for all $x \in R$.

(d) From the following (m,a)[(n,b)(k,c)]=(m,a)(nk,kb+nc+bc)=[mnk,nka+mkb+mnc+mbc+a(kb+nc+bc)] and [(m,a)(n,b)](k,c)=[mnk,kna+kmb+mnc+kab +(na+mb +ac)c] it is clear that A is associative only if R obeys the distributive laws and is associative.

(5.3) Theorem. If R and A are as in theorem (5.1) and $B=\{(o,x)|x \in R\}$ then R and B are isomorphic.

Proof. Obvious.

(5.4) Remark. If R is a RSR then by theorems (5.1), (5.2) and (5.3) R can be embedded in a RSR with a left identity. If R is a SR then R can be embedded in a SR with an identity.

If R is a RSR in which the associative law does not hold and is not a SR then R cannot be embedded in a RR or a LSR since if there did exist an isomorphism of R into a subright ring or a subleft semi-ring it would imply that R is associative or is a SR.

Section 6. Examples

The following examples are given without proof so as to save space. Furthermore $(R,+)$ in the examples will always be an abelian group. The following abreviations will be used. C-commutative wrt $\circ$, A-associative wrt $\circ$, RD-all the elements of the set obey the right distributive law, LD-all the elements of the set obey the left distributive law. If N is placed before any of the above letters is stands for not (e.g. NC means not commutative). Also, the following sets will be used. $I_r=\{x \in R \mid x$ is a right identity$\}$ (I_l), $D_r=\{x \in R \mid x$ is a right divisor of zero$\}$ (D_l), $C_r=\{x \in R \mid x$ obeys the right cancellation law$\}$ (C_l), and $V_r^e=\{x \in R \mid x$ has a right inverse wrt the right (left) identity $e\}$ (V_l^e).

(6.1) Right semi-ring with no identities. Let $(A,+,\circ)$ be any field and $R=A\times A$. For $(x,y),(z,w) \in R$ define $(x,y)+(z,w)=(x+z,y+w)$ and $(x,y)\circ(z,w)=(-xw,yz+x)$.

Then $(R,+,\circ)$ is a RSR such that $\circ$ is NC, NA, and NLD. Furthermore, $I_r=I_l=\emptyset$, $D_r=D_l=R-C_r$, and $C_r=C_l=\{(x,y)\mid x,y\in R$ and $x,y\neq o\}$.

(6.2) Right semi-ring with a right identity. Let $(R,+,\circ)$ be a field. For $x,y\in R$ define $x*y=x+xy$. Then $(R,+,*)$ is a RSR such that $*$ is NC, NA, and NLD. However, $I_r=\{o\}$, $I_l=\emptyset$, $D_r=\{-1\}$, $D_l=R$, $C_r=R-D_r$, $C_l=R-\{o\}$, $V_r^o=D_l$, and $V_l^o=D_r$.

(6.3) Right semi-ring with left identity. Let $(R,+,*)$ be the RSR in example (6.2), and $(\bar{R},+,\circ)$ obtained from theorems (5.1) and (5.2). Then $(\bar{R},+,\circ)$ is a RSR such that $\circ$ is NC, NA, and NLD; and $I_r=\emptyset$, $I_l=\{(1,o)\}$, $D_r=\{(n,-(n+1))\mid n\in I\}\cup\{(o,x)\mid x\neq 1\}$, $D_l=\{(o,x)\mid x\in R\}\cup\{(m,x)\mid x\neq o,\ m\neq o,$ and $x\neq -m\}$, $C_r=R-D_r$, $C_l=\{(m,x)\mid m\neq o$ and $x\neq -m\}$, $V_r^{(1,o)}=\{(m,a)\mid m=\pm 1$ and $a\neq -m\}$ and $V_l^{(1,o)}=\{(1,b)\mid b\neq -(n+1)\}\cup\{(-1,b)\mid b\neq o\}$.

(6.4) Semi-ring with right identity. Let $(A,+,\circ)$ be a field and $R=AxA$. For $(x,y),(z,w)\in R$ define $(x,y)+(z,w)=(x+z,y+w)$ and $(x,y)(z,w)=(-xw,yz)$. Then $(R,+,\circ)$ is a SR such that $\circ$ is NC and NA; and $I_r=\{(1,-1)\}$, $I_l=\emptyset$, $D_r=D_l=\{(x,o)\mid x\in A\}\cup\{(o,x)\mid x\in A\}$, $C_r=C_l=R-D_r$, and $V_r^{(1,-1)}=V_l^{(1,-1)}=\{(x,y)\mid x,y\in A$ and $x,y\neq o\}$.

(6.5) Semi-ring with left identity. Let $(A,+,\circ)$ be a field such that $x^2+y^2=o$ implies $x,y=o$ for $x,y\in A$, and $R=AxA$. For $(x,y),(z,w)\in R$ define $(x,y)+(z,w)=(x+z,y+w)$ and $(x,y)(z,w)=(yz-xw,xz+yw)$. Then $(R,+,\circ)$ is a SR such that $\circ$ is NC and NA; and $I_r=\emptyset$, $I_l=\{(o,1)\}$, $D_r=D_l=\{(o,o)\}$, $C_r=C_l=R-D_r$, and $V_r^{(o,1)}=V_l^{(o,1)}=R-\{(o,o)\}$.

(6.6) Semi-ring with identity. Let $(R,+,\circ)$ be the SR in example (6.5) and $(\bar{R},+,\circ)$ be obtained from theorems (5.1) and (5.2). Then $(\bar{R},+,\circ)$ is a SR such that $\circ$ is NC and NA. Furthermore, $I_r=I_l=\{(1,o)\}$.

(6.7) Right ring with a right identity. Let $(R,+)$ be a commutative group such that $R \supsetneq \{o\}$. If $x,y \in R$ define $x*y=x$. Then $(R,+,*)$ is a RR such that $*$ is NC and NLD. However, $I_r=R$, $I_l=\emptyset$, $D_r=\emptyset$, $D_l=\{o\}$, $C_r=R$, and $C_l=\emptyset$.

(6.8) Right ring with an identity. Let $F=\{f \mid f$ maps the reals into the reals and f is a function$\}$ and for $f,g \in F$ and for an arbitrary $x \in$ reals define $f+g$ and $f*g$ such that $(f+g)(x)=f(x)+g(x)$ and $(f*g)(x)=f(g(x))$. Then $(F,+,*)$ is a RR such that $*$ is NC and NLD. Also, $I_r=I_l=\{I\}$ where I is defined such that for $x \in$ reals $I(x)=x$, $D_r=\{f \in F \mid f$ is not onto$\}$, $D_l=\{f \in F \mid$ there exists an $x \in$ reals such that $f(x)=o\}$, $C_r=F-D_r$, $C_l=F-D_l$, and $V_r^I=V_l^I=\{f \in F \mid f$ is onto and one-to-one$\}$.

(6.9) Ring with no identity. Let R be the set of 2x2 matrices over E(even integers). Then under matrix addition,$+$, and multiplication,$\circ$, $(E,+,\circ)$ is a ring such that $\circ$ is NC. Furthermore, $I_r=I_l=\emptyset$.

(6.10) Ring with right identity. Let $R \subset$ 2x2 matrices over I such that if $\alpha \in R$ then $\alpha=\begin{pmatrix} x & o \\ y & o \end{pmatrix}$ for $x,y \in I$. Then $(R,+,\circ)$ is a ring where $+$ and $\circ$ are matrix addition and multiplication respectively. Furthermore, $\circ$ is NC, $I_r=\left\{\begin{pmatrix} 1 & o \\ x & o \end{pmatrix} \middle| x \in I\right\}$, $I_l=\emptyset$, $D_r=\left\{\begin{pmatrix} o & o \\ x & o \end{pmatrix} \middle| x \in I\right\}$, $D_l=R$, $C_r=\left\{\begin{pmatrix} x & o \\ y & o \end{pmatrix} \middle| x \neq o \text{ and } x,y \in I\right\}$, and $C_l=\emptyset$. The set of right inverses wrt $\cdot$ and wrt the identity $\bar{s}=\begin{pmatrix}1&0\\s&0\end{pmatrix}$ is $I_r^{\bar{s}}=\left\{\begin{pmatrix}x&0\\y&0\end{pmatrix}\middle|\, x=\pm 1 \text{ and } y|s\right\}$ where as the set of left inverses wrt the identity $\bar{s}$ is $I_l^{\bar{s}}=\left\{\begin{pmatrix}x&0\\y&0\end{pmatrix}\middle|\, x=\pm 1\right\}$.

(6.11) Ring with left identity. Let R be a subset of the set of 2x2 matrices over the integers such that if $\alpha \in R$ then $\alpha=\begin{pmatrix}x&y\\0&0\end{pmatrix}$ for $x,y \in I$. Then $(R,+,\cdot)$ is a ring when $+$ and $\cdot$ are the usual matrix addition and multiplication. Furthermore, $I_r=\emptyset$, $I_l=\left\{\begin{pmatrix}1&x\\0&0\end{pmatrix}\middle|\, x\in I\right\}$, $D_r=R$, $D_l=\left\{\begin{pmatrix}0&x\\0&0\end{pmatrix}\middle|\, x\in I\right\}$, $C_r=\emptyset$, $C_l=\left\{\begin{pmatrix}x&y\\0&0\end{pmatrix}\middle|\, x\neq 0,\ x,y\in I\right\}$, $I_r^{\bar{s}}=\left\{\begin{pmatrix}x&y\\0&0\end{pmatrix}\middle|\, x=\pm 1\right\}$, and $I_l^{\bar{s}}=\left\{\begin{pmatrix}x&y\\0&0\end{pmatrix}\middle|\, x=\pm 1 \text{ and } y|s\right\}$ where $\bar{s}=\begin{pmatrix}1&s\\0&0\end{pmatrix}$.

CHAPTER II

M-RINGS

Two definitions of M-Rings will be given with a theorem on their equivalence. M-Subrings, M-Factor rings and M-Ring homomorphisms are defined. As in Chapter I the last section will include the examples.

Section 1. M-Rings and n-multiple M-Rings

(1.1) Definition. If M is a set and $(R,+,\circ)$ is BR then $M-(R,+,\circ)$ is a M-BR wrt η iff $\eta \subset (RxM)xR$ is a function on RxM such that (a) $(x+y,m)\eta=(x,m)\eta+(y,m)\eta$ and (b) $(x\circ y,m)\eta=(x,m)\eta\circ(y,m)\eta$ for any $x,y \in R$ and $m \in M$.

The two conditions (a) and (b) in definition (1.1) are similar to those for a homomorphism. This similarity gives rise to an alternate definition for an M-Ring.

(1.2) Definition. If M is a non-empty set and $(R,+,\circ)$ is a BR then $M-R(R,+,\circ)$ is a M-BR wrt α iff $\alpha \subset MxE(R,+,\circ)$ is a function on M.

(1.3) Theorem. Definition (1.1) and definition (1.2) are equivalent.

Proof. Let $M-(R,+,\circ)$ be an M-BR wrt definition (1.1). For a fixed $m \in M$ and any $x \in R$ define $\bar{m} \subset RxR$ such that $x\bar{m}=(x,m)\eta$. $\bar{m}$ is a function on R since η is a function on RxM. If $x,y \in R$ then $(x+y)\bar{m}=(x+y,m)\eta=(x,m)\eta+(y,m)\eta=x\bar{m}+y\bar{m}$ and $(x\circ y)\bar{m}=(x\circ y,m)\eta=(x,m)\eta\circ(y,m)\eta=x\bar{m}\circ y\bar{m}$. Hence, $\bar{m} \in E(R,+,\circ)$. Now define $\alpha \subset MxE(R)$ by $m\alpha=\bar{m}$ for any $m \in M$.

If $m=m'$ then for any $x \in R$ $x\bar{m}=(x,m)\eta=(x,m')\eta=x\bar{m}'$. Therefore, α is a function on M. Hence, M-(R,+,$\circ$) is an M-BR wrt definition (1.2).

Let M-(R,+,$\circ$) be an M-BR wrt definition (1.2). Denote $m\alpha=\bar{m}$ for $m \in M$. Define $\eta \subset (RxM)xR$ such that for any $x \in R$ and any $m \in M$ $(x,m)\eta=x(m\alpha)=x\bar{m}$. If $(x,m),(x',m') \in RxM$ such that $(x,m)=(x',m')$ then $x=x'$ and $m=m'$, and hence $(x,m)\eta=x(m\alpha)=x'(m'\alpha)=(x',m')\eta$. Therefore η is a function on RxM. For $x,y \in R$ and $m \in M$ $(x+y,m)\eta=(x+y)\bar{m}=x\bar{m}+y\bar{m}=(x,m)\eta+(y,m)\eta$ and $(x\circ y,m)\eta=(x\circ y)\bar{m}=x\bar{m}\circ y\bar{m}=(x,m)\eta\circ(y,m)\eta$. Hence, M-(R,+,$\circ$) is an M-Ring wrt definition (1.1).

(1.4) Remark. From definitions (1.1) and (1.2) an M-BR is dependent upon both η and α. To stress this dependency let M^{α}_{η}-(R,+,$\circ$) denote an M-BR. When no confusion will arise let M-(R,+,$\circ$) or M-R denote an M-BR.

Given a non-empty set and a BR more than one M-BR may be formed.

(1.5) Theorem. If M is a non-empty set and (R,+,$\circ$) is a BR, then there are at least two different M-BRs formed from M and R.

Proof. Let $\alpha \subset MxF$ (F is a non-empty set) be the function on M such that for $x \in R$ and $m \in M$ $x(m\alpha)=o$. For $x,y \in R$ and $m \in M$ $(x+y)(m\alpha)=o=o+o=x(m\alpha)+y(m\alpha)$ and $(x\circ y)(m\alpha)$ $=o=o\circ o=[x(m\alpha)]\circ[y(m\alpha)]$. Therefore, $F \subset E(R)$ and hence, M^{α}_{η}-R is an M-BR where $(x,m)\eta=x(m\alpha)$ for $x \in R$ and $m \in M$.

Define $\alpha' \subset MxF$ (F is a set) such that for $x \in R$ and $m \in M$ $x(m\alpha')=x$. Clearly α' is a function on M and for $x,y \in R$ and $m \in M$ $(x+y)(m\alpha')=x+y=x(m\alpha')+y(m\alpha')$ and

$(x \circ y)(m\alpha') = x \circ y = x(m\alpha') \circ y(m\alpha')$. Hence, $F \subset E(R)$. Therefore, $M^{\alpha'}_{\eta'}$-R is an M-BR where $(x,m)\eta' = x(m\alpha')$ for $x \in R$ and $m \in M$.

(1.6) Remark. To shorten the notation for mappings in an M-BR, M^{α}_{η}-R, we let $xm = (x,m)\eta$. Also denote $M\alpha$ by $\overline{M} \subset E(R)$ and $m\alpha$ for $m \in M$ as $\overline{m}$.

Because more than one M-BR can be otained from a set M and a BR R it is interesting to consider a system consisting of more than one M-BR.

(1.7) Definitions. If M is a non-empty set and $(R,+,\cdot)$ is a BR then $M^{\alpha_1,\ldots,\alpha_n}_{\eta_1,\ldots,\eta_n}$-$(R,+,\circ)$ is an n-multiple M-BR iff $M^{\alpha_i}_{\eta_i}$-$(R,+,\circ)$ for $i=1,\ldots,n$ are M-BRs. If $R^{\alpha_1,\alpha_2}_{\eta_1,\eta_2}$-$(R,+,\circ)$ is a 2-multiple M-BR and $\sqrt{} \subset (RxR)x(RxR)$ is a function on RxR such that $(x,y)\sqrt{} = (y,x)$ for $x,y \in R$ then $R^{\alpha_1,\alpha_2}_{\eta_1,\eta_2}$-$(R,+,\circ)$ is a symmetric 2-multiple M-BR iff $\eta_1 = \sqrt{}\eta_2$.

(1.8) Theorem. If M^{α}_{η}-$(R,+,\circ)$ is an M-BR then for $x \in R$, $m \in M$, $k \in I$ and $j \in N$

(a) $(o,m)\eta = o$

(b) $(kx,m)\eta = k(x,m)\eta$

(c) $(-x,m)\eta = -(x,m)\eta$

(d) $(x^j,m)\eta = [(x,m)\eta]^j$

(e) if $(R,+,\circ)$ is a division ring then

(1) $(x^k,m)\eta = [(x,m)\eta]^k$

(2) $(x^{-1},m)\eta = [(x,m)\eta]^{-1}$

(3) $(1,m)\eta = 1$ when 1 is the identity of R wrt $\circ$ if $m\alpha \neq \overline{o}$

(f) if $(R,+,\circ)$ is a SR with a right (left) identity r wrt $\circ$, R has no left (right) divisors of zero, and $R \neq \{o\}$

then $(r,m)\eta=r$.

Proof. Let $m\alpha=\bar{m}$ for $m\in M$. Then for $x\in R$ $(x,m)\eta=x\bar{m}$.

Part (a). Since $\bar{m}$ is a ring endomorphism $o\bar{m}=o$.

Part (b). Again since $\bar{m}$ is an endomorphism $(kx)\bar{m}=k(x\bar{m})$.

Part (c). Now $o=o\bar{m}=[x+(-x)]\bar{m}=x\bar{m}+(-x)\bar{m}$ which implies $(-x)\bar{m}=-(x\bar{m})$.

Part (d). If $j=1$ then $x^1\bar{m}=(x\bar{m})^1$. Assume true for j. Then $(x^{j+1})\bar{m}=(x^jx^1)\bar{m}=(x^j\bar{m})(x^1\bar{m})=(x\bar{m})^j(x\bar{m})^1=(x\bar{m})^{j+1}$.

Part (e). Since $(R-\{o\},\cdot)$ is a group part (a), (b) and (c) can be applied with notational changes for $x\neq o$. It is clear that (1) and (2) hold for $x=o$.

Part (f). Now $(x,m)\eta r=(x,m)\eta=(xr,m)\eta=(x,m)\eta(r,m)\eta$. By analogous theorems to theorems (1.20) and (1.8) in chapter I all $x\in R$ obey the left cancellation law and $r\neq o$. Hence, $r=(r,m)\eta$.

(1.9) Remark. At first part (d) in theorem (1.8) appears not to need the added condition that R has no zero divisors, but the example presented in section 4 gives an example of a ring with identity 1 that has an endomorphism β on it such that $1\beta\neq 1$. This implies that the identity of a ring is not always mapped onto the identity by an endomorphism.

Section 2. M-Subrings, M-Factor rings, and M-homomorphisms

(2.1) Definition. If M_η^α-R is a M-BR and S is a subBR of R, then M_η^α-S is a M-SubBR of M_η^α-R iff for all

$x \in S$ and $m \in M$ $(x,m)\eta \in S$.

Let $\{S\}$ denote a collection of M-SubBRs of the M-BR M_η^α-R, $\cap S$ denote the intersection of all the $S \in \{S\}$, and $R' = [\cup S]$ to denote a set generated by the finite sums of finite products ($\sum_{i=1}^{k} a_{1_i} \circ \ldots \circ a_{n_i}$, where $a_{j_i} \in S$ for some $S \in \{S\}$).

(2.2) Theorem. If M_η^α-R is a M-BR and $\{S\}$ is a collection of M-SubBRs of R then (a) M_η^α-$\cap S$ is a M-SubBR of M_η^α-R and (b) M_η^α-R' is a M-SubBR of M_η^α-R.

Proof.

Part (a). We need first to show that $\cap S$ is a subBR of R. Let $a,b \in \cap S$ then $a,b \in S$ for all $S \in \{S\}$. Since S is a subBR of R $a-b \in S$ and $a \circ b \in S$ for all $S \in \{S\}$. This implies that $a-b \in \cap S$ and $a \circ b \in \cap S$. Therefore $\cap S$ is a subBR of R. Furthermore because for all $S \in \{S\}$ S is a M-SubBR of M_η^α-R $(a,m)\eta \in S$ for any $a \in S$ and for any $m \in M$. Hence, $(a,m)\eta \in \cap S$ for any $a \in \cap S$ and for any $m \in M$. Therefore M_η^α-$\cap S$ is a M-SubBR of the M-BR M_η^α-R.

Part (b). First let us show that $R' = [\cup S]$ is a subBR of R. For any $x,y \in R'$ $x = \sum_{i=1}^{k} a_{1_i} \circ \ldots \circ a_{1_i}$ and $y = \sum_{i=1}^{q} b_{1_i} \circ \ldots \circ b_{n_i}$. By observation we see that $x-y \in R'$ and $x \circ y \in R'$. Therefore R' is a subBR of R. Letting $(a,m)\eta = am$ then for any $x \in R'$ and for any $m \in M$

$$xm = \left(\sum_{i=1}^{k} a_{1_i} \circ \ldots \circ a_{n_i}\right)m = \sum_{i=1}^{k} (a_{1_i} \circ \ldots \circ a_{n_i})m$$
$$= \sum_{i=1}^{k} (a_{1_i} m \circ \ldots \circ a_{n_i} m)$$

since $a_{i_j} m \in S \in \{S\}$ and by definition of R' $xm = (x,m)\eta \in R'$. Therefore M_η^α-R' is an M-SubBR of the M-BR M_η^α-R.

(2.3) Definitions. If M_η^α-S is an M-SubBR of M_η^α-R then M_η^α-S is a right (left) M-Ideal of M_η^α-R iff S is a right (left) ideal of R. M_η^α-S is an M-Ideal of M_η^α-R iff S is an ideal of R.

(2.4) Theorem. If M_η^α-R is an M-Ring with R being a divison ring then the only left (right) M-Ideals of M_η^α-R are M_η^α-$\{o\}$ and M_η^α-R.

Proof. The only ideals of R are $\{o\}$ and R. Hence the theorem follows from the definition of an M-Ideal.

(2.5) Remark. If M_η^α-R is an M-BR then M_η^α-R has at least two M-Ideals since $\{o\}$ and R are ideals of R and for any $a \in R$ $(a,m)\eta \in R$ and $(o,m)\eta = o \in \{o\}$.

(2.6) Theorem. If M_η^α-$(R,+,\circ)$ is an M-SR, M_η^α-S is an M-Ideal of M_η^α-R, $\overline{R}=R/S$, and $\overline{\eta} \subset (\overline{R}xM)x\overline{R}$ such that $(\overline{x},m)\overline{\eta} = (x+S,m)\overline{\eta} = (x,m)\eta+S$ for all $\overline{x} \in \overline{R}$ and $m \in M$, then $M_{\overline{\eta}}^{\alpha}$-$(\overline{R},\oplus,\circledast)$ is an M-SR called the factor M-SR of M_η^α-R wrt M_η^α-S.

Proof. First we need to prove that $\overline{\eta}$ is a function on $\overline{R}xM$. If $(\overline{a},m)=(\overline{b},m')$ then $\overline{a}=\overline{b}$ and $m=m'$. If $\overline{a}=a+S$ and $\overline{b}=b+S$ then $a=b+i$ for some $i \in S$. Since η is a function and $(a,m)\eta=(b+i,m)\eta=(b,m)\eta+(i,m)\eta$ where $(i,m)\eta \in S$ then $(\overline{a},m)\overline{\eta}=(a,m)\eta+S=(b,m)\eta+(i,m)\eta+S=(b,m)\eta+S=(\overline{b},m)\overline{\eta}$. Therefore $\overline{\eta}$ is a function of $\overline{R}xM$.

By theorem (4.4) in chapter I $(\overline{R},\oplus,\circledast)$ is a SR. For $\overline{a},\overline{b} \in \overline{R}$ and $m \in M$

$$(\overline{a}\oplus\overline{b},m)\overline{\eta}=((a+b)+S,m)\overline{\eta}=(a+b,m)\eta+S=[(a,m)\eta+(b,m)\eta]+S$$
$$=[(a,m)\eta+S]\oplus[(b,m)\eta+S]=(a+S,m)\overline{\eta}\oplus(b+S,m)\overline{\eta}$$
$$=(\overline{a},m)\overline{\eta}\oplus(\overline{b},m)\overline{\eta}$$

$$(\overline{a}\circledast\overline{b},m)\overline{\eta}=(ab+S,m)\overline{\eta}=(ab,m)\eta+S=(a,m)\eta(b,m)\eta+S$$

$=[(a,m)\eta+S]\circledcirc[(b,m)\eta+S]=(a+S,m)\overline{\eta}\circledcirc(b+S,m)\overline{\eta}$

$=(\overline{a},m)\overline{\eta}\circledcirc(\overline{b},m)\overline{\eta}$.

Therefore $M_{\overline{\eta}}^{\overline{\alpha}}$-R is an M-SR.

(2.7) Definition. If M_η^α-R and $M_{\eta'}^{\alpha'}$-R' are M-BRs and $\pi\subset R\times R'$ is a ring-homomorphism then π is an M-BR homomorphism (M-homomorphism) of M_η^α-R into $M_{\eta'}^{\alpha'}$-R' iff for all $x\in R$ and $m\in M$ $[(x,m)\eta]\pi=(x\pi,m)\eta'$.

(2.8) Remarks. Of course we have the special cases of homomorphisms: isomorphisms, endomorphisms and automorphisms.

Note that when M_η^α-R$=M_{\eta'}^{\alpha'}$-R' (ie $\eta=\eta'$, $\alpha=\alpha'$ and R=R') and π is an M-homomorphism of M_η^α-R into $M_{\eta'}^{\alpha'}$-R' then for any $a\in R$ and $m\in M$, where $(a,m)\eta=a(m\alpha)=a\overline{m}$, $a(\overline{m}\pi)=(a\overline{m})\pi=[(a,m)\eta]\pi=(a\pi,m)\eta=(a\pi)\overline{m}=a(\pi\overline{m})$. Hence, the M-endomorphisms of M_η^α-R are just those endomorphisms of R that commute.

(2.9) Notation. When considering homomorphisms of M_η^α-$(R,+,\circ)$ into $M_{\eta'}^{\alpha'}$-$(R',+',\circ')$ no distinction will be made between the different operations. Both + and +' will be denoted by + and both $\cdot$ and $\cdot'$ will be denoted by $\cdot$ (or by no symbol).

Section 3. Fundamental theorems of M-homomorphisms on M-Rings

(3.1) Theorem. If M_η^α-R and $M_{\eta'}^{\alpha'}$-R' are M-BRs and π is an M-homomorphism of M_η^α-R onto $M_{\eta'}^{\alpha'}$-R' then (a) $M_{\eta'}^{\alpha'}$-Rπ is an M-SubBR of $M_{\eta'}^{\alpha'}$-R', and (b) the kernal of π is a left (right) M_η^α-Ideal of M_η^α-R only if R and R' are RSRs (LSRs).

Proof. Part (a). $R\pi$ is a subBR of R' by theorem (4.6) in chapter I. Now for any $a\pi \in R\pi$ and $m \in M$ $(a\pi,m)\eta'=[(a,m)\eta]\pi$ is in $R\pi$. Therefore $M^{\alpha'}_{\eta'}$-$R\pi$ is an M-SubBR of $M^{\alpha'}_{\eta'}$-R'.

Part (b). By theorem (4.7) in chapter I K_π(kernal of π) is a left ideal of the RSR R. For any $a \in K_\pi$ and $m \in M$ $[(a,m)\eta]\pi=(a\pi,m)\eta'=o'$. Therefore, $(a,m)\eta \in K_\pi$ and hence M^{α}_{η}-K is a left M-Ideal of M^{α}_{η}-R.

(3.2) Theorem. If M^{α}_{η}-R, $M^{\alpha'}_{\eta'}$-R' and $M^{\alpha''}_{\eta''}$-R'' are M-BRs, $\pi \subset RxR'$ and $\pi' \subset R'xR''$ are M-homomorphisms then $\pi\pi'$(the resultant) is an M-homomorphism.

Proof. $\pi\pi'$ is a homomorphism of R into R'' by theorem (4.9) in chapter I. For $a \in R$ and $m \in M$ $[(a,m)\eta](\pi\pi')=$ $\{[(a,m)\eta]\pi\}\pi'=[(a\pi,m)\eta']\pi'=((a\pi)\pi',m)\eta''=(a(\pi\pi'),m)\eta''$. Therefore $\pi\pi'$ is an M-homomorphism.

(3.3) Theorem. If $M^{\bar{\alpha}}_{\bar{\eta}}$-$\bar{R}$ is the factor M-SR of the M-SR M^{α}_{η}-R wrt M^{α}_{η}-S and $\tau \subset Rx\bar{R}$ such that $a\tau=a+S$ for $a \in R$ then τ is an M-homomorphism on R (ie $\bar{R}$ is a homomorphic image of R under τ).

Proof. τ is a homomorphism by theorem (4.10) in chapter I. Now by definition $(a+S,m)\bar{\eta}=(a,m)\eta+S$ for $a \in R$ and $m \in M$. Hence, $[(a,m)\eta]\tau=(a,m)\eta+S=(a+S,m)\bar{\eta}=(a\tau,m)\bar{\eta}$. Therefore, τ is an M-homomorphism.

(3.4) Remark. τ as defined in theorem (3.3) is called the natural M-homomorphism of R into $\bar{R}=R/S$.

(3.5) Theorem. If M^{α}_{η}-R and $M^{\alpha'}_{\eta'}$-R' are M-SRs, $\varphi \subset RxR'$ is an M-homomorphism on R, $M^{\bar{\alpha}}_{\bar{\eta}}$-$\bar{R}$ is the factor M-SR of M^{α}_{η}-R wrt M^{α}_{η}-S, and $\bar{\varphi} \subset \bar{R}xR'$ such that $(a+S)\bar{\varphi}=a\varphi$ for all $a \in R$ then (a) $\bar{\varphi}$ is an M-homomorphism of $\bar{R}=R/S$

into R' and $\varphi=\tau\bar{\varphi}$ where τ is the natural M-homomorphism of R into $\bar{R}$ and (b) $\bar{\varphi}$ is an M-isomorphism iff $K_\varphi=S$.

Proof. Part (a). φ is a homomorphism on R into R', $\bar{\varphi}$ is also a homomorphism on $\bar{R}$ into R', and $\varphi=\tau\bar{\varphi}$ by theorem (4.13) in chapter I. For $a\in R$ and $m\in M$ $[(a+S,m)\bar{\eta}]\bar{\varphi}=[(a,m)\eta+S]\bar{\varphi}=[(a,m)\eta]\varphi=(a\varphi,m)\eta'=((a+S)\bar{\varphi},m)\eta'$. Hence, $\bar{\varphi}$ is an M-homomorphism.

Part (b). The result follows directly from part (b) of theorem (4.12) in chapter I.

(3.6) Theorem. If M^{α}_{η}-R and $M^{\alpha'}_{\eta'}$-R' are M-BRs, $\varphi\subset RxR'$ is an M-homomorphism on R onto R', $\{S\}=\{S|S$ is an M-SubBR of R and $K_\varphi\subset S\}$, $\{S'|S'$ is an M-SubBR of R'$\}$, and $\pi\subset\{S\}x\{S'\}$ such that $S\pi=S\tau$ for $S\in\{S\}$ then (a) π is 1-1 and onto (ie $\{S\}\cong\{S'\}$ under π), and (b) S is a right (left) M-Ideal in M-R iff $S'=S\varphi$ is a right (left) M-Ideal in M-R'.

Proof. Part (a). Clearly π is a function on $\{S\}$. Let us prove that π maps $\{S\}$ onto $\{S'\}$. Let $S'\in\{S'\}$. Now define $S=\varphi^{-1}(S')$. For $a,b\in S$ $(a-b)\varphi=a\varphi-b\varphi\in S'$ and $(ab)\varphi=a\varphi b\varphi\in S'$ which implies $a-b\in S$ and $ab\in S$. Therefore S is a subBR of R. For $a\in S$ and $m\in M$ and by definitions $[(a,m)\eta]\varphi=(a\varphi,m)\eta'\in S'$ which implies $(a,m)\eta\in S$ and hence S is a M-SubBR of R. Clearly $S=\varphi^{-1}(S')$ contains $K_\varphi=\varphi^{-1}(o')$ and $S\varphi=S'$. Thus, every M-SubBR of R' can be obtained by applying φ to some M-SubBR of R that contains K_φ. Hence, π is onto. Now let us prove π is 1-1. Let $S\in\{S\}$ and $S_1=\varphi^{-1}(S\varphi)$. Clearly $S_1\supset S$. If $s_1\in S_1$ then $s_1\varphi=s\varphi$ for some $s\in S$. Hence $s_1=s+k$ for some $k\in K_\varphi$. Now since $K_\varphi \subset S$ $s_1 \in S$. Therefore $S=\varphi^{-1}(S\varphi)$. Now if S_1, S_2 $\in \{S\}$ and $S_1\varphi=S_2\varphi$ then $S_1=\varphi^{-1}(S_1\varphi)=\varphi^{-1}(S_2\varphi)=S_2$. Therefore φ is 1-1.

Part (b). Assume S is a right M-Ideal in M-R. Since φ is a homomorphism $(S',+)$ is a subgroup of $(R',+)$. Let $x'=x\varphi \in R'$ and $a'=a\varphi \in S'$. Then $x'a'=(xa)\varphi \in S'$ and hence S' is a right ideal of R'. Furthermore for $a\varphi \in S'$ and $m \in M$ $(a\varphi,m)\eta'=[(a,m)\eta]\varphi \in S$. Hence, S' is a right M-Ideal of M-R'.

Assuming S' to be a right M-Ideal of M-R' we can prove that S is an M-Ideal of R analgous to the above method.

(3.7) Theorem. If M_η^α-R is an M-BR, $\pi \subset (M_\eta^\varphi\text{-}R)x$ $(M_\eta^\alpha\text{-}R)$ is an M-endomorphism, and $R'=\{a \in R \mid a\pi=a\}$ then M_η^α-R' is a M-SubBR of M_η^α-R.

Proof. Since $o\pi=o$ $R'\neq\emptyset$. If $a,b \in R'$ then $(a-b)\pi=a\pi+(-b)\pi=a\pi-b\pi=a-b \in R'$ and $(ab)\pi=a\pi b\pi=ab \in R'$. Hence, R' is a subBR of R by theorem (4.3) in chapter I. For $a \in R'$ and $m \in M$ $[(a,m)\eta]\pi=(a\pi,m)\eta=(a,m)\eta \in R'$. Therefore M_η^α-R' is an M-SubBR of M_η^α-R.

(3.8) Theorem. If M_η^α-R is an M-Ring and R is a division ring then $E(M_\eta^\alpha\text{-}R)-\{\bar{o}\}=A(M_\eta^\alpha\text{-}R)$ where $E(M_\eta^\alpha\text{-}R)$ is the set of M-endomorphisms of M_η^α-R and $A(M_\eta^\alpha\text{-}R)$ is the set of M-automorphisms of M_η^α-R.

Proof. From definition $E(M_\eta^\alpha\text{-}R)-\{\bar{o}\} \supset A(M_\eta^\alpha\text{-}R)$. Let $\bar{m} \in E(M_\eta^\alpha\text{-}R)-\{\bar{o}\}$ and $K_{\bar{m}}$ be the kernel of $\bar{m}$. $K_{\bar{m}}$ is an M-Ideal of M_η^α-R denoted by $M_\eta^\alpha\text{-}K_{\bar{m}}$. Since M_η^α-R is a division M-Ring $M_\eta^\alpha\text{-}K_{\bar{m}}=M_\eta^\alpha\text{-}\{o\}$ or $M_\eta^\alpha\text{-}K_{\bar{m}}=M_\eta^\alpha\text{-}R$. Since $\bar{m}\neq\bar{o}$ and $M_\eta^\alpha\text{-}K_{\bar{m}}\neq$

$M_\eta^\alpha - R\ M_\eta^\alpha - \overline{K_m} = M_\eta^\alpha - \{o\}$. Therefore $E(M_\eta^\alpha - R) - \{\bar{o}\} = A(M_\eta^\alpha - R)$.

Section 4. A particular example

The purpose of this section is to give an example of a ring that has an identity 1 along with an endomorphism β on it such that $1\beta \neq 1$, and hence give a counter example to the pseudo-theorem if $M_\eta^\alpha - R$ is an M-Ring and R has an identity then for $m \in M$ $(1,m)\eta = 1$.

Let R be the following ring:

+	o	1	a	a+1
o	o	1	a	a+1
1	1	o	a+1	a
a	a	a+1	o	1
a+1	a+1	a	1	o

•	o	1	a	a+1
o	o	o	o	o
1	o	1	a	a+1
a	o	a	a	o
a+1	o	a+1	o	a+1

Clearly the following subset R' of R is a subring of R.

+	o	a
o	o	a
a	a	o

•	o	a
o	o	o
a	o	a

Define $\beta \subset R \times R'$ such that $o\beta = o$, $1\beta = a$, $a\beta = a$ and $(a+1)\beta = o$.

Now we need to check to see if β is an endomorphism. If $x,y \in R$ then for

(1) $x=o$. $(x+y)\beta=y\beta=o+y\beta=o\beta+y\beta=x\beta+y\beta$ and $(xy)\beta=o\beta=o=y\beta=o\beta y\beta=x\beta y\beta$

(2) $x=1$, $y=a$. $(1+a)\beta=o=a+a=1\beta+a\beta$ and $(1a)\beta=a\beta=a=aa=1\beta a\beta$

(3) $x=1$, $y=a+1$. $(1+a+1)\beta=a\beta=a=a+o=1\beta+(a+1)\beta$ and $[1(a+1)]\beta=(a+1)\beta=o=1\beta o=1\beta(a+1)\beta$

(4) $x=1$, $y=1$. $(1+1)\beta=o\beta=o=o+o=1\beta+1\beta$ and $(11)\beta=1\beta=a=aa=1\beta 1\beta$

(5) $x=a$, $y=a$. $(a+a)\beta=o\beta=o=a+a=a\beta+a\beta$ and $(aa)\beta=a\beta=a=aa=a\beta a\beta$

(6) $x=a$, $y=a+1$. $(a+a+1)\beta=1\beta=a=a+o=1\beta+(a+1)\beta$ and $[a(a+1)]\beta=o\beta=o=a\beta o=a\beta(a+1)\beta$.

Hence, β is an endomorphism such that $1\beta=a\neq 1$.

Let $M=\{2\}$ and define α such that $2\alpha=\beta$. Then M^{α}_{η}-R is an M-Ring.

Section 5. Examples of M-Rings

(5.1) Example of an M-Ring. Let $(R,+,\cdot)$ be any ring. Define $\eta\subset[RxE(R)]xR$ such that $(a,\overline{m})\eta=a\overline{m}$ for $a\in R$ and $\overline{m}\in E(R)$. The α associated with η is the identity map of $E(R)$ onto $E(R)$. If $(a,\overline{m})=(a',\overline{m'})$ then $a=a'$ and $\overline{m}=\overline{m'}$, and $(a,\overline{m})\eta=a\overline{m}=a'\overline{m'}=(a',\overline{m'})\eta$. Hence, η is a function on $RxE(R)$. For $a,b\in R$ and $\overline{m}\in E(R)$ $(a+b,\overline{m})\eta=(a+b)\overline{m}=a\overline{m}+b\overline{m}=(a,\overline{m})\eta+(b,\overline{m})\eta$ and $(ab,\overline{m})\eta=(ab)\overline{m}=a\overline{m}b\overline{m}=(a,\overline{m})\eta(b,\overline{m})\eta$.

Therefore $E(R)^{\alpha}_{\eta}$-R is an M-Ring.

(5.2) Example of an M-Ring. Let C denote the set of complex numbers, and $R=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; a,b,c,d \in C\right\}$ and define $+$ and $\circ$ as the usual matrix addition and multiplication on R. Then $(R,+,\circ)$ is a non-commutative ring. Let $S=\left\{\begin{pmatrix} a & b \\ o & c \end{pmatrix} \middle| \; a,b,c \in C\right\} \subset R$. Then for $\alpha=\begin{pmatrix} a & b \\ o & c \end{pmatrix}$, $\beta=\begin{pmatrix} a' & b' \\ o & c' \end{pmatrix} \in S$

$$\alpha\beta=\begin{pmatrix} a & b \\ o & c \end{pmatrix}\begin{pmatrix} a' & b' \\ o & c' \end{pmatrix}=\begin{pmatrix} aa' & ab'+bc' \\ o & cc' \end{pmatrix} \in S$$

and

$$\alpha+(-\beta)=\begin{pmatrix} a & b \\ o & c \end{pmatrix}+\begin{pmatrix} -a' & -b' \\ o & -c' \end{pmatrix}=\begin{pmatrix} a-a' & b-b' \\ o & c-c' \end{pmatrix} \in S.$$

Hence, S is a subring of R. Define the map $\bar{m}_{r_1}$ such that $\alpha\bar{m}_{r_1}=\begin{pmatrix} a & o \\ o & o \end{pmatrix}$. Clearly $\bar{m}_{r_1}$ is single-valued. Also,

$$(\alpha\beta)\bar{m}_{r_1}=\begin{pmatrix} aa' & o \\ o & o \end{pmatrix}=\begin{pmatrix} a & o \\ o & o \end{pmatrix}\begin{pmatrix} a' & o \\ o & o \end{pmatrix}=\alpha\bar{m}_{r_1}\beta\bar{m}_{r_1}$$

and

$$(\alpha+\beta)\bar{m}_{r_1}=\begin{pmatrix} a+a' & o \\ o & o \end{pmatrix}=\begin{pmatrix} a & o \\ o & o \end{pmatrix}+\begin{pmatrix} a' & o \\ o & o \end{pmatrix}=\alpha\bar{m}_{r_1}+\beta\bar{m}_{r_1}.$$

Therefore $\bar{m}_{r_1} \in E(S)$. Define $\bar{m}_{r_2},\ldots,\bar{m}_{r_8}$ such that $\alpha\bar{m}_{r_2}=\begin{pmatrix} o & o \\ o & c \end{pmatrix}$, $\alpha\bar{m}_{r_3}=\begin{pmatrix} a & o \\ o & b \end{pmatrix}$, $\alpha\bar{m}_{r_4}=\begin{pmatrix} o & o \\ o & o \end{pmatrix}$, $\alpha\bar{m}_{r_5}=\begin{pmatrix} 1 & o \\ o & 1 \end{pmatrix}$, $\alpha\bar{m}_{r_6}=\begin{pmatrix} 1 & o \\ o & o \end{pmatrix}$, $\alpha\bar{m}_{r_7}=\begin{pmatrix} a & b \\ o & c \end{pmatrix}$, and $\alpha\bar{m}_{r_8}=\begin{pmatrix} o & o \\ o & 1 \end{pmatrix}$. It can be shown that all of the above defined maps are in E(S). Consider now the set $M=\{1,\ldots,8\}$ and the map π such that $i\pi=\bar{m}_{r_i}$ for $i=1,\ldots,8$. By the definition of π it is clear that it is a single-valued map of M into E(S). Hence M and S along with π form an M-Ring.

(5.3) Example of an M-Ring. Let R and S be the same rings as in example (5.2). For $a \in C$ let $\bar{a}$ denote the conjugate of a. Define the maps $\bar{m}_{1_1},\ldots,\bar{m}_{1_8}$ such that for $\alpha\in S$ $\alpha\bar{m}_{1_1}=\begin{pmatrix}\bar{a} & o\\ o & o\end{pmatrix}$, $\alpha\bar{m}_{1_2}=\begin{pmatrix}o & o\\ o & \bar{c}\end{pmatrix}$, $\alpha\bar{m}_{1_3}=\begin{pmatrix}\bar{a} & o\\ o & \bar{b}\end{pmatrix}$, $\alpha\bar{m}_{1_4}=\begin{pmatrix}o & o\\ o & o\end{pmatrix}$, $\alpha\bar{m}_{1_5}=\begin{pmatrix}1 & o\\ o & 1\end{pmatrix}$, $\alpha\bar{m}_{1_6}=\begin{pmatrix}1 & o\\ o & o\end{pmatrix}$, $\alpha\bar{m}_{1_7}=\begin{pmatrix}\bar{a} & \bar{b}\\ o & \bar{c}\end{pmatrix}$, $\alpha\bar{m}_{1_8}=\begin{pmatrix}o & o\\ o & 1\end{pmatrix}$. Clearly $\bar{m}_{1_1}$ is single-valued. For $\alpha=\begin{pmatrix}a & b\\ o & c\end{pmatrix}$, $\beta=\begin{pmatrix}a' & b'\\ o & c'\end{pmatrix}\in S$

$$(\alpha\beta)\bar{m}_{1_1}=\begin{pmatrix}\overline{aa'} & o\\ o & o\end{pmatrix}=\begin{pmatrix}\bar{a}\bar{a}' & o\\ o & o\end{pmatrix}=\begin{pmatrix}\bar{a} & o\\ o & o\end{pmatrix}\begin{pmatrix}\bar{a}' & o\\ o & o\end{pmatrix}=\alpha\bar{m}_{1_1}\beta\bar{m}_{1_1}$$

and

$$(\alpha+\beta)\bar{m}_{1_1}=\begin{pmatrix}\overline{a+a'} & o\\ o & o\end{pmatrix}=\begin{pmatrix}\bar{a}+\bar{a}' & o\\ o & o\end{pmatrix}=\begin{pmatrix}\bar{a} & o\\ o & o\end{pmatrix}\begin{pmatrix}\bar{a}' & o\\ o & o\end{pmatrix}=\alpha\bar{m}_{1_1}\beta\bar{m}_{1_1}$$

Hence, $\bar{m}_{1_1}\in E(S)$. Similarly it can be shown that the rest of the maps are in E(S) also. Define $\tau\subset M\times E(S)$ ($M=\{1,\ldots,8\}$) such that $n\tau=\bar{m}_{1_n}$ for $n=1,\ldots,8$. By the definition of τ is is clear that it is a single-valued map on M into E(S). Hence, M and S along with τ form an M-Ring.

(5.4) Example of a 2-multiple M-Ring. By considering examples (5.2) and (5.3) together we have a 2-multiple M-Ring. Note that if we denote the images of $\alpha\bar{m}_{r_n}$ and $\alpha\bar{m}_{1_n}$ as $\alpha\#n$ and $n\#\alpha$ respectively then $4\#\alpha=\alpha\#4$, $5\#\alpha=\alpha\#5$, $6\#\alpha=\alpha\#6$ and $8\#\alpha=\alpha\#8$ are the only ones that commute.

(5.5) Example of an M-Ring. Let $(F[x],+,\cdot)$ be the extension of the field F over x. Then for $a,b,c\in F[x]$ $a=\sum_{i=o}^{n_a}a_ix^i$, $b=\sum_{i=o}^{n_b}b_ix^i$, and $c=\sum_{i=o}^{n_c}c_ix^i$. Define $\bar{m}_u\subset F[x]\times F[x]$ such that $a\bar{m}_u=\sum_{i=o}^{n_a}a_iu^i$ where $u\in F[x]$. If a=b then $n_a=n_b$ and $a_i=b_i$ for $i=1,\ldots,n_a$. Hence $a\bar{m}_u=\sum_{i=o}^{n_a}a_iu^i=\sum_{i=o}^{n_b}b_iu^i=b\bar{m}_u$. Therefore $\bar{m}_u$ is a function on F[x]. Furthermore,

$$(a+b)\bar{m}_u=\left(\sum_{i=o}^{\max(n_a,n_b)}(a_i+b_i)x^i\right)\bar{m}_u=\sum_{i=o}^{\max(n_a,n_b)}(a_i+b_i)u^i$$

$$=\sum_{i=o}^{\max(n_a,n_b)}a_iu^i+\sum_{i=o}^{\max(n_a,n_b)}b_iu^i=\sum_{i=o}^{n_a}a_iu^i+\sum_{i=o}^{n_b}b_iu^i=a\bar{m}_u+b\bar{m}_u$$

since $a_i=o$ for $i>n_a$ and $b_i=o$ for $i>n_b$, and

$$(ab)\bar{m}_u=\left(\sum_{i=o}^{n_a+n_b}\left(\sum_{j=o}^{i}a_jb_{i-j}\right)x^i\right)\bar{m}_u=\sum_{i=o}^{n_a+n_b}\left(\sum_{j=o}^{i}a_jb_{i-j}\right)u^i$$

$$=\left(\sum_{i=o}^{n_a}a_iu^i\right)\left(\sum_{i=o}^{n_b}b_iu^i\right)=(a\bar{m}_u)(b\bar{m}_u).$$

Now define $\alpha\subset F[x]xE(F[x])$ such that $u\alpha=\bar{m}_u$ for $u\in F[x]$.

If $u=u'$ then $a\bar{m}_u=\sum_{i=o}^{n_a}a_iu^i=\sum_{i=o}^{n_a}a_i(u')^i=a\bar{m}_{u'}$. Therefore α is a function on $F[x]$. Defining $a(u\alpha)=(a,u)\eta$ for $a,u\in F[x]$ $F[x]_\eta^\alpha-F[x]$ is an M-Ring.

(5.6) Example of an M-Ring. If in the example above we had required $u\in F$ then $M=F$ and $F_\eta^\alpha-F[x]$ would be an M-Ring.

(5.7) Example of an M-Subring. Consider the M-Ring in example (5.5). F is a subring of $F[x]$ and for $a\in F$ and $\bar{m}_u\in E(F[x])$ $a\bar{m}_u=a\in F$. Hence, $F[x]_\eta^\alpha-F$ is a M-Subring of $F[x]_\eta^\alpha-F[x]$.

(5.8) Example of an M-Subring. Consider the M-Ring in example (5.6). Clearly $F_\eta^\alpha-F$ is an M-Subring of $F_\eta^\alpha-F[x]$.

(5.9) Example of an M-homomorphism. On the M-Rings $F[x]_\eta^\alpha-F[x]$ and $F[x]_\eta^\alpha-F$ define $\pi\subset F[x]xF$ such that for a in F[x] $a\pi=a_o$ where $a=\sum_{i=o}^{n_a} a_i x^i$. If a=b then $a\pi=a_o=b_o=b\pi$ and hence π is a function on F[x]. For $a,b\in F[x]$ $(a+b)\pi=a_o+b_o=a\pi+b\pi$ and $(ab)\pi=a_ob_o=a\pi b\pi$. Hence π is a homomorphism of F[x] into F. For $a\in F[x]$ $(a\bar{m}_u)\pi=a_o=a\pi=(a\pi)\bar{m}_u$. Hence π is an M-homomorphism of F[x] into F.

(5.10) Example of an M-Ring. Let $R=\{(a_1,\ldots,a_n) \mid a_i\in \text{reals for all } i\}$ and define + and $\cdot$ such that for $a=(a_1,\ldots,a_n)$, $b=(b_1,\ldots,b_n)\in R$ $a+b=(a_1+b_1,\ldots,a_n+b_n)$ and $a\cdot b=(a_1\cdot b_1,\ldots,a_n\cdot b_n)$. Clearly $(R,+,\cdot)$ is a ring. Define the map $\bar{m}_i$ such that $a\bar{m}_i=(a_{i_1},\ldots,a_{i_n})$ where $(i_1,\ldots,i_n)$ is a permutation of $(1,\ldots,n)$. Clearly $\bar{m}_i$ is a function on R. For $a,b\in R$ $(a+b)\bar{m}_i=(a_1+b_1,\ldots,a_n+b_n)\bar{m}_i=(a_{i_1}+b_{i_1},\ldots,a_{i_n}+b_{i_n})=(a_{i_1},\ldots,a_{i_n})+(b_{i_1},\ldots,b_{i_n})=a\bar{m}_i+b\bar{m}_i$ and $(ab)\bar{m}_i=(a_{i_1}b_{i_1},\ldots,a_{i_n}b_{i_n})=(a_{i_1},\ldots,a_{i_n})(b_{i_1},\ldots,b_{i_n})=(a\bar{m}_i)(b\bar{m}_i)$. Hence $\bar{m}_i\in E(R)$. Now define $\alpha\subset M\times E(R)$ such that $i\alpha=\bar{m}_i$ where $M=\{1,\ldots,n!\}$. By the definition of α it is a function on M. Therefore $M_\eta^\alpha\text{-}R$ for $(a,i)\eta=a(i\alpha)=a\bar{m}_i$ is an M-Ring.

(5.11) Example of an M-Ring where M is the set R. Let R be any division ring. Define $\varphi_b\subset R\times R$ such that for $a\in R$ $a\varphi_b=bab^{-1}$ where $b\in R$. If $a=a'$ then $ba=ba'$ and $bab^{-1}=ba'b^{-1}$. Hence φ_b is function on R for any $o\neq b\in R$. For $a,c\in R$ $(a+c)\varphi_b=b(a+c)b^{-1}=bab^{-1}+bcb^{-1}=a\varphi_b+b\varphi_b$ and $(ac)\varphi_b=b(ac)b^{-1}=b[a(b^{-1}b)c]b^{-1}=(bab^{-1})(bcb^{-1})=(a\varphi_b)(c\varphi_b)$. Therefore $\varphi_b\in E(R)$ for each $b\in R$. Define $\alpha\subset R\times E(R)$ such that $b\alpha=\varphi_b$. If $b=b'$ then $b^{-1}=(b')^{-1}$ and $a\varphi_b=bab^{-1}=b'a(b')^{-1}$ $=a\varphi_{b'}$ for all $a\in R$. Therefore α is a function on R. Hence for $\eta\subset(RxR)xR$ such that $(a,b)\eta=a(b\alpha)$ for $a,b\in R$ R_η^α-R is an M-Ring.

Note that if R is a comutative division ring then $a\varphi_b$ $=a=a\varphi_c$ for all $b,c\in R$ and hence $\varphi_b=\varphi_c$ for all $b,c\in R$. Furthermore the mapping φ_b is just the identity map for all $b\in R$.

(5.12) Example of an M-Subring. Let R_η^α-R be the M-Ring in example (5.11). Let $R'=\{a\in R|ab=ba$ for all $b\in R\}$. $R'\neq\emptyset$ since $1,o\in R'$. If $a,b\in R'$ then for any $c\in R$ $ac=ca$ and $bc=cb$. Thus $(ab)c=a(bc)=a(cb)=(ac)b=(ca)b=c(ab)$ and $(a-b)c=ac-bc=ca-cb=c(a-b)$. Therefore $ab,a-b\in R'$ and hence R' is a subring of R. For $a\in R'$ and $\varphi_b\in E(R)$ $a\varphi_b=$ $bab^{-1}=a(bb^{-1})=a\in R'$. Hence R_η^α-R' is an M-Subring of R_η^α-R.

(5.13) Example of an M-Ring. Let $F=\{f|f$ is a function that maps the reals into the reals$\}$. By defining + and $\cdot$ such that for any $f,g\in F$ and $x\in$ reals $(f+g)(x)=$ $f(x)+g(x)$ and $(f\cdot g)(x)=f(x)\cdot g(x)$ $(F,+,\cdot)$ is a ring. Define $\eta\subset(FxF)xF$ such that $(f,g)\eta=f\#g$ where for $x\in R$ $[f\#g](x)=f(g(x))$. For $(f,g)=(f',g')$ $[f\#g](x)=f(g(x))=$ $f(g'(x))=f'(g'(x))=[f'\#g'](x)$. Therefore η is a function on FxF. Futhermore for $f,g,h\in F$ and $x\in R$ $[(f+g)\#h](x)=$ $(f+g)(h(x))=f(h(x))+g(h(x))=[f\#h](x)+[g\#h](x)=[(f\#h)+$ $(g\#h)](x)$ and $[(fg)\#h](x)=(fg)(h(x))=f(h(x))g(h(x))=$ $[f\#h](x)[g\#h](x)=[(f\#h)(g\#h)](x)$. Therefore F_η^α-F is an M-Ring where $(f,g)\eta=f(g\alpha)$ for $f,g\in F$.

(5.14) Example of an M-Ring. Let R be a commutative ring of characteristic p where p is a prime. Define $\bar{m} \subset R \times R$ such that for $a \in R$ $a\bar{m}=a^p \in R$. Then if $a=a'$ $a^p=(a')^p$. Therefore $\bar{m}$ is a function on R. Also for $a,b \in R$ $(a+b)\bar{m}=(a+b)^p=a^p+b^p=a\bar{m}+b\bar{m}$ and $(ab)\bar{m}=(ab)^p=a^p b^p=a\bar{m}b\bar{m}$. Hence $\bar{m} \in E(R)$. Define $\alpha_1 \subset R \times E(R)$ such that for all $a \in R$ $a\alpha_1=\bar{m}$ and $(a,b)\eta=a(b\alpha_1)=a\bar{m}$. Hence, $R_{\eta}^{\alpha_1}$-R is an M-Ring.

We can obtain another M-Ring from the above example by defining $\alpha_2 \subset \{o,1,2\} \times E(R)$ such that $o\alpha_2=\bar{o} \in E(R)$, $1\alpha_2=\bar{1} \in E(R)$ [$\bar{1}$ is the identity endomorphism] and $2\alpha_2=\bar{m}$. Then $\{o,1,2\}_{\eta}^{\alpha}$-R is an M-Ring.

CHAPTER III

EXPANDED RINGS

The right expanded ring will be defined and its equivalence with an M-Ring proved. The question as to whether an expanded ring exists will be resolved. Again the last sections will be reserved for examples.

Section 1. Right semi-expanded rings

(1.1) Definitions. $(R,+,\circ,\#)$ is a bare expanded BR (BEBR) iff $(R,+,\circ)$ is a BR and # is a binary operation on R. If $(R,+,\circ,\#)$ is a BEBR then R is a right semi-expanded BR (RSEBR) iff for any $x,y,z \in R$ (a) $(x+y)\#z=(x\#z)+(y\#z)$ and (b) $(x\circ y)\#z=(x\#z)\circ(y\#z)$.

(1.2) Notation. We adopt the convention that when $x\#y\circ z\#w$ or $x\#y+z\#w$ are written we mean $(x\#y)\circ(z\#w)$ and $(x\#y)+(z\#w)$ respectively.

(1.3) Theorem. If $(R,+,\circ,\#)$ is a RSEBR and $\#=\eta$ then $R^{\alpha}_{\eta}-(R,+,\circ)$ is an M-BR.

Proof. η is a function on RxR since # is a binary operation on R. For $x,y,z \in R$ $(x+y,z)\eta=(x+y)\#z=x\#z+y\#z=(x,z)\eta+(y,z)\eta$ and $(x\circ y,z)\eta=(x\circ y)\#z=x\#z\circ y\#z=(x,z)\eta\circ(y,z)\eta$.

(1.4) Theorem. If $R^{\alpha}_{\eta}-(R,+,\circ)$ is an M-BR and $\#=\eta$ then $(R,+,\circ,\#)$ is a RSEBR.

Proof. Same as in theorem (1.3).

(1.5) Remark. From theorems (1.3) and (1.4) a RSEBR is simply an M-BR where M is equal to the set R.

(1.6) Theorem. If $(R,+,\circ,\#)$ is a RSEBR then for $x,y \in R$, $k \in I$, and $j \in N$

(a) $o\#x=o$

(b) $(kx)\#y=k(x\#y)$

(c) $(-x)\#y=-(x\#y)$

(d) $x^j\#y=(x\#y)^j$

(e) if $(R,+,\cdot)$ is a SR with a right (left) identity r wrt $\cdot$, R has no left (right) divisors of zero, and $R \neq \{o\}$ then $r\#x=r$.

(f) if $(R,+,\circ)$ is a division ring then

(1) $x^k\#y=(x\#y)^k$

(2) $x^{-1}\#y=(x\#y)^{-1}$

(3) $1\#x=1$ where 1 is the identity of R wrt $\cdot$ and $x\alpha$ is not equal to the zero-endomorphism of R.

Proof. Follows from theorem (1.8) in chapter II.

(1.7) Remark. If $(R,+,\cdot,\#)$ is a BEBR we can consider $(R,+,\#)$ as a BR, and if $(R,+,\circ,\#)$ is a RSEBR $(R,+,\#)$ can be considered as a RSR. Hence, all of the theorems in chapter I section I can be applied to $(R,+,\#)$ with notational changes. However, many of the theorems and definitions will be restated to give more continuity to this section.

(1.8) Definitions. If $(R,+,\circ,\#)$ is a BEBR then $r_\#$ is a right (left) identity of R wrt # iff for any $x \in R$ $x\#r_\#=x$ $(r_\#\#x=x)$. R has a two sided identity (or identity)

$r_{\#}$ wrt # iff $r_{\#}$ is both a left and a right identity of R wrt #.

(1.9) Theorem. If R is a RSEBR with a left identity $r_{\#}$ wrt # and $R \neq \{o\}$ then $r_{\#} \neq o$.

Proof. Same as in theorem (1.8) in chapter I.

(1.10) Theorem. If $(R,+,\circ,\#)$ is a RSESR with a left identity $r_{\#}$ wrt #, $(R,+,\circ)$ has a right (left) identity r wrt $\circ$ and $(R,+,\bullet)$ has no left (right) divisors of zero, then $r_{\#} \notin \{o,r\}$.

Proof. Theorem (1.9) shows $r_{\#} \neq o$. Assume $r_{\#}=r$. Then $o=r_{\#}\#o=r\#o=r$. But this contradicts theorem (1.8) in chapter I.

(1.11) Remark. If R is a RSEBR which has a right identity $r_{\#}$ wrt # then $r_{\#}$ may be zero as shown by example (5.1). Furthermore, if R is a RSEBR which has a right (left) identity r wrt $\bullet$ and has a right identity $r_{\#}$ wrt # then $r_{\#}$ may be equal to r as shown in example (5.3).

(1.12) Theorem. If R is a RSEBR with R^{α}_{η}-R as its M-Ring equivalence and R has a left identity e wrt # but R has no right identities wrt # then (a) α is one to one and (b) $I \notin R\alpha=\overline{R}$ where for $x \in R$ $xI=x$.

Proof. Same as in theorem (1.13) in chapter I.

(1.13) Definitions. If R is a BEBR then $x \in R$ is a right (left) divisor of $z \in R$ wrt # iff there exists a $y \in R$ such that $y \neq z$ and $y\#x=z$ $(x\#y=z)$. $x \in R$ is a general right (left) divisor of z wrt # iff there exists a $y \in R$ such that $y\#x=z$ $(x\#y=z)$. $x \in R$ is a divisor of z (general divisor of z) wrt # iff x is both a right and a left divisor of z (x is both a general right and a general left divisor of z).

(1.14) Remark. Our attention will be focused on divisors of zero and divisors of identities wrt $\cdot$. In example (6.3) b+l is a general divisor of o, 1, a, and a+l as well as a divisor of o, l, a, and a+l wrt #. Also, in example (6.3) we see that b+l is a divisor of a+b+l wrt $\cdot$, but b+l is not a divisor of a+b+l wrt #. Furthermore, in example (6.3) b+l is a divisor of a+l wrt #, but b+l is not a divisor of a+l wrt $\cdot$. Hence there seems to be no connection between divisors wrt # and divisors wrt $\cdot$ for arbitrary divisors.

(1.15) Definitions. If R is a BEBR then $x \in R$ obeys right cancellation law wrt # iff when $y,z \in R$ such that y#x=z#x then y=z; $x \in R$ obeys the left concellation law wrt # iff when $y,z \in R$ such that x#y=x#z then y=z.

(1.16) Theorem. If R is a RSEdivision ring with l as the identity wrt $\cdot$ and $x \in R$ obeys the right cancellation law wrt # then x is not a right divisor of zero or l wrt #, and x is a general right divisor of l only if $\bar{x} \neq \bar{o}$.

Proof. If there exists $y \in R$ such that y#x=o then by theorem (1.6) y#x=o#x=o. Since x obeys the right cancellation law y=o. Hence, x is not a right divisor of zero. If there exists a $y \in R$ such that y#x=l then by theorem (1.6) y#x=l#x=l. Which implies y=l because x obeys the right cancellation law. Hence, x is not a right divisor of 1. If $\bar{x}=\bar{o}$ then by theorem (1.6) $1\#x=1$.

(1.17) Theorem. If R is a RSE division ring with 1 as its identity wrt $\cdot$, $x \in R$ is not a right divisor of zero wrt #, and x is not a right divisor of 1 wrt # then x obeys the right cancellation law.

Proof. Let $y,z \in R$ such that $y\#x=z\#x$. If $y=o$ then $z=o$ since x is not a right divisor of zero wrt #. Also, if $z=o$ then $y=o$. So x obeys the right cancellation law in these cases. Suppose $y,z \neq o$. Then $y\#x, z\#x \neq o$. Because $(R,+,\cdot)$ is a division ring $z\#x$ has an inverse wrt $\cdot$, and by theorem (1.6) $(z\#x)^{-1}=z^{-1}\#x$. Hence, $1=(y\#x)(z\#x)^{-1}=(y\#x)(z^{-1}\#x)=(yz^{-1})\#x$. Since x is not a right divisor of 1 wrt # $yz^{-1}=1$ which implies $y=z$.

(1.18) Remark. From theorems (1.16) and (1.17) if $D_{rq}=\{x \in R \mid x$ is a right divisor of q wrt $\#\}$ and $C_r=\{x \in R \mid$ x obeys the right cancellation law wrt $\#\}$ then $C_r=(R-D_{ro}) \cap (R-D_{r1})=R-(D_{ro} \cup D_{r1})$.

(1.19) Definition. If R is a BEBR and R has a right (left) identity r wrt # then y is a right inverse of x wrt # relative to r iff $x\#y=r$, and y is a left inverse of x wrt # relative to r iff $y\#x=r$.

Section 2. Semi-expanded rings

In this section we are going to define the left semi-BR and the semi-BR. Then the rest of the section is devoted to deriving an equivalence theorem that will be helpful in finding an example of a semi-ring.

(2.1) Definitions. If $(R,+,\circ,\#)$ is a BEBR then R is a left semi-expanded BR (LSEBR) iff for any $x,y,z \in R$ (a) $z\#(x+y)=z\#x+z\#y$ and (b) $z\#(xy)=(z\#x)(z\#y)$; and R is a semi-expanded BR (SEBR) iff R is a RSEBR and a LSEBR.

(2.2) Remark. A LSEBR is really the same as a RSEBR except for the notation. Hence, all of the properties of RSEBRs apply to LSEBRs.

(2.3) Notation. If $(R,+,\circ,\#)$ is a BEBR let $\#_1=+$, $\#_2=\circ$, and $\#_3=\#$.

(2.4) Theorem. If $(R,+,\circ,\#)$ is a SEBR, $\eta_1=\#$, and $\eta_2=\varphi\eta_1$ where $\varphi \subset (RxR)x(RxR)$ is a function on RxR such that $(x,y)\varphi=(y,x)$ for $x,y \in R$ then $R^{\alpha_1,\alpha_2}_{\eta_1,\eta_2}-(R,+,\circ)$ is a symmetric 2-multiple M-BR.

Proof. A symmetric 2-multiple M-BR was defined in definition (1.7) of chapter II.

Clearly η_1 and η_2 are both functions on RxR into R. For $x,y,z \in R$ and $i=1,2$

$$(x\#_i y,z)\eta_1=(x\#_i y)\#z=(x\#z)\#_i(y\#z)=(x,z)\eta_1\#_i(y,z)\eta_1$$

and $(x\#_i y,z)\eta_2=(x\#_i y,z)\varphi\eta_1=(z,x\#_i y)\eta_1=z\#(x\#_i y)=$
$(z\#x)\#_i(z\#y)=(z,x)\eta_1\#_i(z,y)\eta_1=(x,z)\varphi\eta_1\#_i(y,z)\varphi\eta_1=$
$(x,z)\eta_2\#_i(y,z)\eta_2$.

(2.5) Theorem. If $R^{\alpha_1,\alpha_2}_{\eta_1,\eta_2}-(R,+,\circ)$ is a symmetric 2-multiple M-BR and $\#=\eta_1$ then $(R,+,\cdot,\#)$ is a SEBR.

Proof. Since η_1 is a function on RxR # is a binary operation on R. For $x,y,z \in R$ and $i=1,2$

$$(x\#_i y)\#z=(x\#_i y,z)\eta_1=(x,z)\eta_1\#_i(y,z)\eta_1=(x\#z)\#_i(y\#z)$$

and $z\#(x\#_i y)=(z,x\#_i y)\eta_1=(z,x\#_i y)\varphi\eta_2=(x\#_i y,z)\eta_2=$ $(x,z)\eta_2\#_i(y,z)\eta_2=(x,z)\varphi\eta_1\#_i(y,z)\varphi\eta_1=(z,x)\eta_1\#_i(z,y)\eta_1=$ $(z\#x)\#_i(z\#y)$.

(2.6) Remarks. From theorems (2.4) and (2.5) a SEBR can be considered as a symmetric 2-multiple M-BR and a special type of a symmetric 2-multiple M-BR can be considered as a SEBR. The theorems for RSEBRs and LSEBRs hold for SEBRs.

(2.7) Theorem. If $R^{\alpha_1,\alpha_2}_{\eta_1,\eta_2}-(R,+,\cdot)$ is the symmetric 2-multiple M-BR that is equivalent with the SEBR $(R,+,\cdot,\#)$ then for $x,y,z\in R$ $(z,x\#_i y)\eta_1=(z,x)\eta_1\#_i(z,y)\eta_1$ for $i=1,2$.

Proof. Let $x,y,z\in R$. Then $(z,x\#_i y)\eta_1=(z,x\#_i y)\varphi\eta_2=$ $(x\#_i y,z)\eta_2=(x,z)\eta_2\#_i(y,z)\eta_2=(x,z)\varphi\eta_1\#_i(y,z)\varphi\eta_1=$ $(z,x)\eta_1\#_i(z,y)\eta_1$.

(2.8) Definition. If $(R,+,\cdot)$ is a BR and $E(R)$ is the set of all endomorphisms on R then we define $\bar{+}$ and $\bar{\cdot}$ subsets of $[E(R)xE(R)]xF$ where $F\subset RxR$ such that if $z\in R$ and $\beta,\delta\in E(R)$ then $z(\beta,\delta)\bar{+}=z\beta+z\delta$ and $z(\beta,\delta)\bar{\cdot}=z\beta\cdot z\delta$.

(2.9) Theorem. If $(R,+,\cdot)$ is a BR, and $\bar{+}$ and $\bar{\cdot}$ are defined as in definition (2.8) then (a) $F=\{f \mid f$ is a function on R into R$\}$ and (b) $\bar{+}$ and $\bar{\cdot}$ are functions.

Proof. Let $\tau=(\beta,\delta)$, $\tau'=(\beta',\delta')\in E(R)xE(R)$.

Part (a). Let $z=w\in R$. Then for $\bar{\#}_1=\bar{+}$, $\bar{\#}_2=\bar{\cdot}$, $\#_1=+$, $\#_2=\cdot$, and $i=1,2$

$$z(\tau\bar{\#}_i)=z((\beta,\delta)\bar{\#}_i)=z\beta\#_i z\delta=w\beta\#_i w\delta=w((\beta,\delta)\bar{\#}_i)=w(\tau\bar{\#}_i).$$

Hence, $\tau\overline{\#}_i$ for $i=1,2$ is a function on R. Therefore, $F=\{f \mid f$ is a function on R into R$\}$.

Part (b). Let $z \in R$ and $\tau=\tau'$ which implies $\beta=\beta'$ and $\delta=\delta'$. Then for $i=1,2$

$$z(\tau\overline{\#}_i)=z((\beta,\delta)\overline{\#}_i)=z\beta\#_i z\delta=z\beta'\#_i z\delta'=z((\beta',\delta')\overline{\#}_i)=$$
$$z(\tau\overline{\#}_i).$$

Since z was arbitrary $\tau\overline{\#}_i=\tau'\overline{\#}_i$ for $i=1,2$. Hence, $\overline{\#}_1=\overline{+}$ and $\overline{\#}_2=\overline{\circ}$ are functions.

(2.10) Remark. As shown in theorem (2.9) $\beta \overline{+} \delta$ and $\beta \overline{\circ} \delta$ are both functions on R into R for $\beta,\delta \in E(R)$, but for an arbitrary BR, R, neither $\beta \overline{+} \delta$ nor $\beta \overline{\circ} \delta$ need be an endomorphism on R.

(2.11) Theorem. If $(R,+,\circ,\#)$ is a SEBR with $R^{\alpha_1,\alpha_2}_{\eta_1,\eta_2}-(R,+,\circ)$ as its symmetric 2-multiple M-BR equivalent, and $\overline{R}=R\alpha_1 \subset E(R)$ then $(\overline{R},\overline{+},\overline{\circ})$ is a BR where $\overline{+}$ and $\overline{\circ}$ are defined as in definition (2.8), and α_1 is a homomorphism of R onto $\overline{R}$.

Proof. From theorem (2.9) $\overline{+}$ and $\overline{\circ}$ are functions. If $\overline{a},\overline{b} \in \overline{R}$ then there exists $a,b \in R$ such that $a\alpha_1=\overline{a}$ and $b\alpha_1=\overline{b}$. Let $x \in R$. Then by the definition of $x\alpha_1$ and theorem (2.7)

$$x\overline{(a\#_i b)}=x[(a\#_i b)\alpha_1]=(x,a\#_i b)\eta_1=(x,a)\eta_1\#_i(x,b)\eta_1=$$
$$x(a\alpha_1)\#_i x(b\alpha_1)=x(\overline{a}\ \overline{\#}_i\ \overline{b}) \text{ for } i=1,2.$$

Since x was arbitrary $\overline{(a\#_i b)}=\overline{a}\ \overline{\#}_i\ \overline{b}$ for $i=1,2$. Thus for $\overline{a},\overline{b} \in \overline{R}$ $\overline{a}\ \overline{\#}_i\ \overline{b}=\overline{(a\#_i b)} \in \overline{R}$. So, $\overline{+}$ and $\overline{\circ}$ restricted to

$\bar{R}x\bar{R}$ are binary operations on $\bar{R}$. Furthermore, since $(a\#_i b)\alpha_1=\overline{(a\#_i b)}=\bar{a}\ \bar{\#}_i\ \bar{b}=(a\alpha_1)\ \bar{\#}_i\ (b\alpha_1)$ for i=1,2 α_1 is a homomorphism of R onto $\bar{R}$. Thus, $\bar{R}$ is a BR under $\bar{+}$ and $\bar{\circ}$.

(2.12) Theorem. If $(R,+,\circ)$ is a BR, $(S,\bar{+},\bar{\circ})$ is a BR of endomorphisms of R wrt $\bar{+}$ and $\bar{\circ}$ as defined in definition (2.8), and α is a homomorphism of R into S then $(R,+,\circ,\#)$ is a SEBR where $x\#y=x(y\alpha)$ for $x,y\in R$.

Proof. By definition $\#\subset(RxR)xR$. Let $(a,b)=(a',b')$ which implies $a=a'$ and $b=b'$. Then

$$(a,b)\#=a\#b=a(b\alpha)=a'(b'\alpha)=a'\#b'=(a',b')\#.$$

Therefore, # is a function on RxR and hence a binary operation on R. Let $x,y,z\in R$. Then for i=1,2

$$(x\#_i y)\#z=(x\#_i y)(z\alpha)=[x(z\alpha)]\#_i[y(z\alpha)]=(x\#z)\#_i(y\#z).$$

Since α is a homomorphism

$$z\#(x\#_i y)=z[(x\#_i y)\alpha]=z[(x\alpha)\bar{\#}_i(y\alpha)]=[z(x\alpha)]\#_i[z(y\alpha)]$$

$$=(z\#x)\#_i(z\#y) \text{ for } i=1,2.$$

Therefore, $(R,+,\circ,\#)$ is a SEBR.

(2.13) Remark. As shown in section 6 there does exist rings that satisfies the hypothesis of theorem (2.12), and hence there exists SEBRs.

(2.1) Theorem. If $(R,+,\circ,\#)$ is a SESR with $R^{\alpha_1,\alpha_2}_{\eta_1,\eta_2}-(R,+,\circ)$ as its symmetric 2-multiple M-BR equivalent, $\bar{R}=R\alpha_1$, and $x\#y=x(y\alpha_1)=x\bar{y}$ for $x,y\in R$ then for $x,y,a,b\in R$ (a) $a\bar{x}b\bar{y}+a\bar{y}b\bar{x}=o$ (b) $a\bar{x}b\bar{y}+b\bar{x}a\bar{y}=o$ (c) $a\bar{y}b\bar{x}=b\bar{x}a\bar{y}$ (d) $(ab+ab)\bar{x}=o$ (e) $(ab+ba)\bar{x}=o$ (f) $(a^2+a^2)\bar{x}=o$

(g) $\bar{x} \mathbin{\bar{\circ}} \bar{y}=\bar{y} \mathbin{\bar{\circ}} \bar{x}$ (h) $(ab)\bar{x}=(ba)\bar{x}$ (i) $a\bar{o}=o$ for all $a \in R$ (j) $(a\bar{x})^2+(b\bar{x})^2=(a\bar{x}+b\bar{x})^2$ (k) $(a\bar{x}-b\bar{x})^2=(a\bar{x}+b\bar{x})^2$ (l) $\bar{x} \mathbin{\bar{\circ}} \bar{y} \mathbin{\bar{+}} \bar{y} \mathbin{\bar{\circ}} \bar{x}=\bar{o}$ and (m) $\bar{x} \mathbin{\bar{\circ}} \bar{y} \mathbin{\bar{+}} \bar{x} \mathbin{\bar{\circ}} \bar{y}=\bar{o}$.

Proof. Let $x,y,a,b \in R$ and $\bar{x}=a\alpha$, $\bar{y}=y\alpha \in \bar{R}$. Then since $\overline{x+y}=\bar{x} \mathbin{\bar{+}} \bar{y}$ and $\overline{x \circ y}=\bar{x} \mathbin{\bar{\circ}} \bar{y}$ are endomorphisms

(a) $(ab)\overline{(x+y)}=a\overline{(x+y)}b\overline{(x+y)}=a(\bar{x} \mathbin{\bar{+}} \bar{y})b(\bar{x} \mathbin{\bar{+}} \bar{y})=$ $(a\bar{x}+a\bar{y})(b\bar{x}+b\bar{y})=a\bar{x}b\bar{x}+a\bar{x}b\bar{y}+a\bar{y}b\bar{x}+a\bar{y}b\bar{y}$ and $(ab)\overline{(x+y)}=$ $(ab)\bar{x}+(ab)\bar{y}=a\bar{x}b\bar{x}+a\bar{y}b\bar{y}$. Therefore, $a\bar{x}b\bar{y}+a\bar{y}b\bar{x}=o$.

(b) $(a+b)\overline{(xy)}=(a+b)(\bar{x} \mathbin{\bar{\circ}} \bar{y})=(a+b)\bar{x}(a+b)\bar{y}=$ $(a\bar{x}+b\bar{x})(a\bar{y}+b\bar{y})=a\bar{x}a\bar{y}+a\bar{x}b\bar{y}+b\bar{x}a\bar{y}+b\bar{x}b\bar{y}$ and $(a+b)\overline{(xy)}=$ $a(\bar{x} \mathbin{\bar{\circ}} \bar{y})+b(\bar{x} \mathbin{\bar{\circ}} \bar{y})=a\bar{x}a\bar{y}+b\bar{x}b\bar{y}$. Therefore, $a\bar{x}b\bar{y}+b\bar{x}a\bar{y}=o$.

(c) From (a) and (b) $a\bar{y}b\bar{x}=b\bar{x}a\bar{y}$.

(d) Let $y=x$ in (a).

(e) Let $y=x$ in (b).

(f) Let $y=x$ and $a=b$ in (a).

(g) If $a=b$ in (c) then $a(\bar{y} \mathbin{\bar{\circ}} \bar{x})=a(\bar{x} \mathbin{\bar{\circ}} \bar{y})$. Since a was arbitrary $\bar{y} \mathbin{\bar{\circ}} \bar{x}=\bar{x} \mathbin{\bar{\circ}} \bar{y}$.

(h) Let $x=y$ in (c).

(i) Let $a \in R$. Then $a\bar{o}=a\overline{(x-x)}=a[\bar{x} \mathbin{\bar{+}} \overline{(-x)}]=$ $a\bar{x}+a\overline{(-x)}$. By the analogous theorem of theorem (1.6) for LSEBRs and since R is a RSring $x\#(-y)=-(x\#y)$. Thus, $a\overline{(-x)}=a\#(-x)=-(a\#x)=-[a(x\alpha)]=-(a\bar{x})$. Hence, $a\bar{o}=o$.

(j) Since $(a\bar{x})^2=a^2\bar{x}$ $(a\bar{x}+b\bar{x})^2=a^2\bar{x}+b^2\bar{x}+a\bar{x}b\bar{x}+b\bar{x}a\bar{x}=$ $a^2\bar{x}+b^2\bar{x}=(a\bar{x})^2+(b\bar{x})^2$.

(k) Similar to (j).

(l) By (i) and (a) if $a=b$ and arbitrary $a(\bar{x} \mathbin{\bar{\circ}} \bar{y} \mathbin{\bar{+}} \bar{x} \mathbin{\bar{\circ}} \bar{y})=o=a\bar{o}$. Since a was arbitrary the theorem follows.

(m) Similar to (l).

(2.15) Theorem. If R satisfies the same conditions as in theorem (2.14) and R has a right identity r wrt $\circ$ then (a) $\bar{a} \mathbin{\bar{\circ}} \bar{r}=\bar{a}$ (b) $\bar{o} \mathbin{\bar{\circ}} \bar{a}=\bar{o}$, and (c) $\bar{a} \mathbin{\bar{+}} \bar{a}=\bar{o}$ for all $\bar{a}$ $\bar{R}$.

Proof. For $a \in R$ (a) follows since $\bar{a}=\overline{a \circ r}=\bar{a} \mathbin{\bar{\circ}} \bar{r}$. Since $o \circ a=o$ for a SR $\bar{o}=\overline{o \circ a}=\bar{o} \mathbin{\bar{\circ}} \bar{a}$. In theorem (2.14) part (m) let $y=r$ and $a=x$. Then from part (a) above $\bar{a} \mathbin{\bar{\circ}} \bar{r}+\bar{a} \mathbin{\bar{\circ}} \bar{r}=\bar{o}$ implies $\bar{a}+\bar{a}=\bar{o}$.

(2.16) Theorem. If R satisfies the same conditions as in theorem (2.14) and R is not commutative wrt $\cdot$ then the identity map of R onto R is not in $\bar{R}$.

Proof. If $I \in \bar{R}$, where $I \subset R \times R$ such that $xI=x$ for all $x \in R$, then by part (c) in theorem (2.14) $ab=aIbI=bIaI=ba$ for all $a,b \in R$.

(2.17) Theorem. If R is the same as in theorem (2.14) and every non-zero element of R has order different than two wrt $(R,+,\cdot)$ then $\bar{a} \mathbin{\bar{\cdot}} \bar{b}=\overline{a \cdot b}=\bar{o}$.

Proof. Assume there exists $a,b \in R$ such that $\bar{a} \mathbin{\bar{\cdot}} \bar{b} \neq \bar{o}$. Then there exists $c \in R$ such that $c\overline{(a \cdot b)} \neq o$. By part (l) of theorem (2.14) $c[\overline{(a \cdot b)} \mathbin{\bar{+}} \overline{(a \cdot b)}]=c\bar{o}=o$. Hence, $o \neq c\overline{(a \circ b)} \in R$ has order two.

(2.18) Theorem. If R is the same as in theorem (2.17) and R has a right (left) identity then $a \# b=o$ for all $a,b \in R$.

Proof. Follows from theorems (2.15) and (2.17).

(2.19) Theorem. If R is an integral domain with characteristic different than two then there doesnot exist a binary operation # on R such that $(R,+,\circ,\#)$ is not a trivial SEring (a trivial SEring is one where $a\#b=o$ for all $a,b \in R$).

Proof. Follows directly from theorem (2.18).

(2.20) Theorem. If $(R,+,\circ)$ is a division ring and $\{o,1\} \subseteq R$ where 1 is the identity of R wrt $\circ$ then there doesnot exist a binary operation # such that $(R,+,\circ,\#)$ is a not a trivial SEring.

Proof. Assume such a binary operation # exists. From theorem (2.15) part (a) $\overline{1} \mathbin{\bar{\circ}} \overline{1}=\overline{1}$. Hence, for any $o \neq x \in R$ $x(\overline{1} \mathbin{\bar{\circ}} \overline{1})=(x\overline{1})\circ(x\overline{1})=x\overline{1}$ which implies that $x\overline{1}=1$. If there doesnot exist an $o \neq x \in R$ such that $x+1 \neq o$ then for all $o \neq x \in R$ $x+1=o$. Thus, $x=-1$ for all $x \neq o$. Hence, $x=-1=1$ which implies $R=\{o,1\}$. Therefore, there exists a $o \neq x \in R$ such that $x+1 \neq o$. Now $1=(x+1)\overline{1}=x\overline{1}+1\overline{1}=1+1$ which implies that $1=o$. But by theorem (1.8) in chapter I $1 \neq o$.

<u>Section 3. Right expanded rings</u>

(3.1) Definition. If $(R,+,\circ,\#)$ is a RSEBR then R is a right expanded BR (REBR) iff the operation # is associative. If R is a SEBR then R is an expanded BR iff the operation # is associative.

(3.2) Remark. Most of the theorems about RSEBRs hold for REBRs.

(3.3) Theorem. If R is a BEBR with a right identity r wrt # and x has a right inverse wrt # then x is not a right divisor of zero wrt #.

Proof. Apply theorem (3.3) in chapter I.

(3.4) Theorem. If $(R,+,\circ,\#)$ is a REdivision ring with a right identity r wrt # and x has a right inverse wrt # then x is not a right divisor of 1 wrt # where 1 is the identity wrt $\cdot$.

Proof. Let y be the right inverse of x wrt # relative to r. Assume x is a right divisor of 1. Then there exists a $z \in R$ such that $z \neq 1$ and $z\#x=1$. Using theorem (1.6) $z=z\#r=z\#(x\#y)=(z\#x)\#y=1\#y=1$.

Section 4. Ideals and homomorphisms

(4.1) Notation. When BEBR is used in the same sentence several times it refers to the same type of BEBR under discussion.

(4.2) Definitions. If $(R,+,\cdot,\#)$ is a BEBR and $\emptyset \neq S \subset R$ then $(S,+,\cdot.\#)$ is a subBEBR iff $(S,+,\circ,\#)$ is a BEBR. $\emptyset \neq S \subset R$ is a right (left) ideal of the BEBR R iff $(S,+,\cdot)$ is a subBR and $x\#s \in S$ $(s\#x \in S)$ for all $x \in R$ and $s \in S$. $\emptyset \neq S \subset R$ is a right-right (right-left) ideal of the BEBR R iff S is a right (left) ideal of the BR $(R,+,\circ)$ and S is a right ideal of the BEBR $(R,+,\circ,\#)$. Similarly, left-left and left-right ideals of a BEBR R are defined. $\emptyset \neq S \subset R$ is a ideal of the BEBR R iff S is a left ideal of the BEBR R and S is a right ideal of the BEBR R. $\emptyset \neq S \subset R$ is a dual ideal of the BEBR iff S is an ideal of the BEBR $(R,+,\circ,\#)$ and S is an ideal of the BR $(R,+,\circ)$.

(4.3) Theorem. If R is a BEBR and $\emptyset \neq S \subset R$ then S is a subBEBR iff for $x,y \in R$ $x-y \in R$, $x \circ y \in R$ and $x\#y \in R$.

Proof. Obvious.

(4.4) Theorem. If $(R,+,\circ,\#)$ is a SESR, $\emptyset \neq S \subset R$ is a dual ideal of R and $\oplus,\odot,\oplus\!\!\!\!\#\subset (R/S\mathrm{x}R/S)\mathrm{x}R/S$ such that for $x=a+S, y=b+S \in R/S$ $(x,y)\oplus=(a+b)+S$, $(x,y)\odot=a\circ b+S$, and $(x,y)\oplus\!\!\!\!\#=a\#b+S$ then $(R/S,\oplus,\odot,\oplus\!\!\!\!\#)$ is a SESR with $\odot$ and $\oplus\!\!\!\!\#$ being associative iff $\circ$ and $\#$ are associative respectively.

Proof. From the definitions of dual ideals and SEBRs, and from theorem (4.4) in chapter I $(R/S,\oplus,\odot)$ and $(R/S,\oplus,\oplus\!\!\!\!\#)$ are SRs with $\odot$ and $\oplus\!\!\!\!\#$ being associative iff $\circ$ and $\#$ are associative respectively. For $x=a+S$, $y=b+S, z=c+S \in R/S$

$(x\odot y)\oplus\!\!\!\!\#z=(ab+S)\oplus\!\!\!\!\#z=(ab)\#c+S=(a\#c)(b\#c)+S=$ $(a\#c+S)\odot(b\#c+S)=[(a+S)\oplus\!\!\!\!\#(c+S)]\odot[(b+S)\oplus\!\!\!\!\#(c+S)]=(x\oplus\!\!\!\!\#z)\odot(y\oplus\!\!\!\!\#z)$ and $z\oplus\!\!\!\!\#(x\odot y)=z\oplus\!\!\!\!\#(ab+S)=c\#(ab)+S=(c\#a)(c\#b)+S=$ $(c\#a+S)\odot(c\#b+S)=[(c+S)\oplus\!\!\!\!\#(a+S)]\odot[(c+S)\oplus\!\!\!\!\#(b+S)]=(z\oplus\!\!\!\!\#x)\odot(z\oplus\!\!\!\!\#y)$. Hence, $(R/S,\oplus,\odot,\oplus\!\!\!\!\#)$ is a SESR.

(4.5) Theorem. If η is a homomorphism of the BEBR $(R,+,\circ,\#)$ into the BEBR $(\bar{R},\bar{+},\bar{\circ},\bar{\#})$ then $R\eta \subset \bar{R}$ is a subBEBR.

Proof. By theorm (4.6) in chapter I $(R\eta,\bar{+},\bar{\circ})$ is a subBR of $(\bar{R},\bar{+},\bar{\circ})$. For $x=a\eta, y=b\eta \in R\eta$ $x\bar{\#}y=(a\eta)\bar{\#}(b\eta)=$ $(a\#b)\eta \in R\eta$. Hence, by theorem (4.3) $(R\eta,\bar{+},\bar{\circ},\bar{\#})$ is a subBEBR of $\bar{R}$.

(4.6) Remark. One might wonder if there are two BEBRs with a homomorphism between them as defined in the definition (0.3). Examples (7.3) and (7.4) shows that they do exist.

(4.7) Theorem. If η is a homomorphism of the RSEBR $(R,+,\circ,\#)$ into the RSEBR $(\bar{R},\bar{+},\bar{\circ},\bar{\#})$ and $(R,+,\circ)$ and $(\bar{R},\bar{+},\bar{\circ})$ are RSRs (LSRs) then K_η (kernal of η) is a left-left (left-right) ideal of the RSERSR (RSELSR) R.

Proof. By theorem (4.7) in chapter I K_η is a left ideal of $(R,+,\circ)$. Let $x \in R$ and $a \in K_\eta$. Then $(a\#x)\eta=(a\eta)\bar{\#}(x\eta)=\bar{o}\ \bar{\#}\ (x\eta)=\bar{o}$. Therefore, $a\#x \in K_\eta$ and hence K_η is a left-left ideal of $(R,+,\circ,\#)$.

(4.8) Remark. A similar theorem to (4.7) holds for LSEBRs.

(4.9) Theorem. If η is a homomorphism of the SESR $(R,+,\circ,\#)$ into the SESR $(\bar{R},\bar{+},\bar{\circ},\bar{\#})$ then K_η is a dual ideal of R.

Proof. Follows from theorem (4.8) in chapter I, theorem (4.7) above and remark (4.8) above.

(4.10) Theorem. If $(R,+,\circ,\#)$, $(\bar{R},\bar{+},\bar{\circ},\bar{\#})$ and $(\bar{\bar{R}},\bar{\bar{+}},\bar{\bar{\circ}},\bar{\bar{\#}})$ are BEBRs, and $\varphi \subset R\times\bar{R}$ and $\bar{\varphi} \subset \bar{R}\times\bar{\bar{R}}$ are homomorphisms then $\varphi\bar{\varphi}$ (the resultant) is a homomorphism of R into $\bar{\bar{R}}$.

Proof. Similar to the proof of theorem (4.9) in chapter I.

(4.11) Theorem. If $(\bar{R},\oplus,\odot,\circledast)$ is the factor SESR of the SESR $(R,+,\cdot,\#)$ wrt the dual ideal S of R and $\beta \subset R\times\overline{R}$ such that for $x\in R$ $x\beta=x+S$ then β is a homomorphism.

Proof. If $x,y\in R$ then $(x\#y)\beta=(x\#y)+S=(x+S)\oplus(y+S)=x\beta\oplus y\beta$. Hence, by theorem (4.10) in chapter I β is a homomorphism of R into $\overline{R}$.

(4.12) Remark. β as defined in theorem (4.11) is called the natural homomorphism of R into $\overline{R}=R/S$.

(4.13) Theorem. If R and R' are SESRs, $\varphi\subset R\times R'$ is a homomorphism, $\overline{R}$ is the factor SESR of R wrt the dual ideal $\emptyset\neq S\subset K_{\varphi}$, and $\overline{\varphi}\subset\overline{R}\times R'$ such that $(x+S)\overline{\varphi}=x\varphi$ for all $x\in R$ then (a) $\overline{\varphi}$ is a homomorphism of $\overline{R}=R/S$ into R' and $\varphi=\beta\overline{\varphi}$ where β is the natural homomorphism of R into $\overline{R}$ and (b) $\overline{\varphi}$ is an isomorphism iff $K_{\varphi}=S$.

Proof. Along with the proof of theorem (4.12) in chapter I and with the fact that for $x+S,y+S\in\overline{R}$ $[(x+S)\oplus(y+S)]\overline{\varphi}=(x\#y+S)\overline{\varphi}=(x\#y)\varphi=(x\varphi)\#(y\varphi)=(x+S)\overline{\varphi}\#(y+S)\overline{\varphi}$ we have part (a). Part (b) follows from part (b) of theorem (4.12) in chapter I.

(4.14) Remark. Note that there is a difference between an M-homomorphism and a homomorphism. As shown in example (7.3) they may be the same, but need not be as shown in example (7.4).

Section 5. Examples of RSEBRs

The following examples are given without proof so as to save space. Furthermore, $(R,+)$ in the examples will always be an abelian group. The following abreviations will be used. C-commutative, A-associative, RD-obeys the right distributive law, LD- obeys the left distributive law. If N is placed before any of the above abreviations it stands for not (e.g. NC means not commutative). Also, the following sets are used. $I_r = \{x \in R \mid x \text{ is a right identity wrt } \#\}$ (I_l), $D_{rz} = \{x \in R \mid x \text{ is a right divisor of } z \text{ wrt } \#\}$ (D_{lz}), $C_r = \{x \in R \mid x \text{ obeys the right cancellation law wrt } \#\}$ (C_l), and $V_r^e = \{x \in R \mid x \text{ has a right inverse wrt } \# \text{ relative to the right (left) identity } e\}$ (V_l^e).

(5.1) REBR R with right identities but no left identities. Let R be a BR with at least two elements. Define $x\#y=x$ for all $x,y \in R$. Then $(R,+,\circ,\#)$ is a REBR such that # is NC and NLD. However, $I_r=R$, $I_l=\emptyset$, $D_{ro}=\emptyset$, $D_{lo}=\{o\}$, $C_r=R$ and $C_l=\emptyset$. If $(R,+,\cdot)$ is a division ring then $D_{r1}=\emptyset$ and $D_{l1}=\{1\}$.

(5.2) RER with a unique right identity. Let R be the M-Ring in example (5.5) in chapter II. Define # as $a\#b=\sum_{i=o}^{n_a} a_i(b)^i$ where $a=\sum_{i=o}^{n_a} a_i x^i$, $b=\sum_{i=o}^{n_b} b_i x^i \in F[x]$. Then $(R,+,\circ,\#)$ is a RER. # is NC and NLD. $I_r=\{x\}$, $I_l=\emptyset$, $D_{ro}=D_{r1}=\{a \in F[x] \mid a \in F\}$, and $C_l=\{a \in F[x] \mid a \notin F\}$. The sets C_r, D_{lo}, and D_{l1} are not easily found.

(5.3) RSE division ring with a unique right identity. Let R be the M-Ring in example (5.11) in chapter

II. For # defined as $x\#y=x\varphi y=yxy^{-1}$ $(R,+,\circ,\#)$ is a RSE division ring where # is NC, NA, and NLD. $I_r=\{1\}$, $I_l=\emptyset$, $D_{ro}=\{o\}$, $D_{lo}=\{o\}$, $D_{rl}=\emptyset$, $D_{ll}=\{1\}$, $C_r=R$, C_l is not easily determined, $V_r^1=R$, and $V_l^1=1$.

Notice that the identity wrt $\cdot$ is the same as the right identity wrt #.

(5.4) RER with an identity. Let R be the M-Ring in example (5.13) in chapter II. Then $(R,+,\circ,\#)$ is a RER where # is NC and NLD. $I_r=I_l=\{I\}$ where $I(x)=x$ for all $x\in$ reals, $D_{ro}=\{f\in R \mid f \text{ is not onto}\}$ and $D_{lo}=\{f\in R \mid$ there exists an $x\in$ reals such that $f(x)=o\}$. 1 such that $1(x)=1$ for all $x\in$ reals is the identity of $(R,+,\cdot)$ wrt $\cdot$. $D_{rl}=\{f\in R \mid f \text{ is not onto}\}$, $D_{ll}=\{f\in R \mid$ there exists an $x\in$ reals such that $f(x)=1\}$, $C_r=\{f\in R \mid f \text{ is onto}\}$, $C_l=\{f\in R \mid f \text{ is one-to-one}\}$, and $V_r^I=V_l^I=\{f\in R \mid f$ is one-to-one and onto$\}$.

Section 6. Examples of SErings

Using the equivalence theorems and conditions on a SEBR examples of SErings will be given.

(6.1) Trivial SBR (semi-BR). Let $(R,+,\cdot)$ be a BR such that $o\cdot o=o$. Define $\#\subset(R\times R)\times R$ such that for $x,y\in R$ $x\#y=o$. Clearly, # is a binary operation. Furthermore, for $x,y,z\in R$ $(x+y)\#z=o=o+o=x\#z+y\#z$, $(xy)\#z=o=o\cdot o=(x\#z)(y\#z)$, and $x\#y=o=y\#x$. Hence, $(R,+,\circ,\#)$ is a semi-ring.

(6.2) Example of an expanded ring that is commutative wrt the third operation and an example of a SEring that is not commutative wrt to the third operation.

Let R be the following ring.

+	o	1	a	a+1
o	o	1	a	a+1
1	1	o	a+1	a
a	a	a+1	o	1
a+1	a+1	a	1	o

$\cdot$	o	1	a	a+1
o	o	o	o	o
1	o	1	a	a+1
a	o	a	a	o
a+1	o	a+1	o	a+1

The following are subrings of R.

R_1:

+	o
o	o

$\cdot$	o
o	o

R_2:

+	o	1
o	o	1
1	1	o

$\cdot$	o	1
o	o	o
1	o	1

R_3:

+	o	a
o	o	a
a	a	o

$\circ$	o	a
o	o	o
a	o	a

R_4:

+	o	a+1
o	o	a+1
a+1	a+1	o

$\circ$	o	a+1
o	o	o
a+1	o	a+1

Define

(a) $\bar{o} \subset RxR_1$ such that $xo=o$ for all $x \in R$

(b) $\varphi_2 \subset RxR_2$ such that $o\varphi_2=o=a\varphi_2$ and $1\varphi_2=1=(a+1)\varphi_2$

(c) $\varphi_3 \subset RxR_3$ such that $o\varphi_3=o=(a+1)\varphi_3$ and $1\varphi_3=a=a\varphi_3$

(d) $\varphi_4 \subset RxR_4$ such that $o\varphi_4=o=a\varphi_4$ and $1\varphi_4=a+1=(a+1)\varphi_4$.

Clearly $\bar{o}$, φ_2, φ_3, and φ_4 are functions. If $x,y \in R$ then $(x+y)\bar{o}=o=o+o=x\bar{o}+y\bar{o}$ and $(xy)\bar{o}=o=oo=(x\bar{o})(y\bar{o})$. Hence $\bar{o} \in E(R)$. Let us show that φ_2, φ_3, and φ_4 are also in $E(R)$. If $x,y \in R$ then for $i=2,3,4$

$$(x+x)\varphi_i=o\varphi_i=o=x\varphi_i+x\varphi_i,\ (xx)\varphi_i=(x\varphi_i)(x\varphi_i)=x\varphi_i$$

and for .

(a) $x=o$, $y\neq o$

$(x+y)\varphi_i=y\varphi_i=o+y\varphi_i=x\varphi_i+y\varphi_i$

$(xy)\varphi_i=o\varphi_i=o=oy\varphi_i=(x\varphi_i)(y\varphi_i)$.

(b) $x=1$, $y=a$

$(1+a)\varphi_2=1=1\varphi_2+a\varphi_2$, $(1+a)\varphi_3=o=1\varphi_3+a\varphi_3$,

$(1+a)\varphi_4=a+1=1\varphi_4+a\varphi_4$, $(1a)\varphi_2=a\varphi_2=o=(1\varphi_2)(a\varphi_2)$,

$(1a)\varphi_3=a\varphi_3=a=(1\varphi_3)(a\varphi_3)$, $(1a)\varphi_4=o=(1\varphi_4)(a\varphi_4)$.

(c) $x=1$, $y=a+1$

$(1+a+1)\varphi_2=a\varphi_2=o=1\varphi_2+(a+1)\varphi_2$, $(1+a+1)\varphi_3=a\varphi_3=$

$a=1\varphi_3+(a+1)\varphi_3$, $(1+a+1)\varphi_4=a\varphi_4=o=1\varphi_4+(a+1)\varphi_4$,

$[1(a+1)]\varphi_2=(a+1)\varphi_2=1=(1\varphi_2)[(a+1)\varphi_2]$, $[1(a+1)]\varphi_3$

$=(a+1)\varphi_3=o=(1\varphi_3)[(a+1)\varphi_3]$, $[1(a+1)]\varphi_4=(a+1)\varphi_4$

$=a+1=(1\varphi_4)[(a+1)\varphi_4]$.

(d) $x=a$, $y=a+1$

$(a+a+1)\varphi_2=1\varphi_2=1=a\varphi_2+(a+1)\varphi_2$, $(a+a+1)\varphi_3=1\varphi_3=$

$a=a\varphi_3+(a+1)\varphi_3$, $(a+a+1)\varphi_4=1\varphi_4=o=a\varphi_4+(a+1)\varphi_4$,

$[a(a+1)]\varphi_2=o\varphi_2=o=(a\varphi_2)[(a+1)\varphi_2]$, $[a(a+1)]\varphi_3=$

$o\varphi_3=o=(a\varphi_3)[(a+1)\varphi_3]$, $[a(a+1)]\varphi_4=o\varphi_4=o=$

$(a\varphi_4)[(a+1)\varphi_4]$.

Hence, φ_2, φ_3, and φ_4 are also in $E(R)$.

Let us show now that $\{\bar{o},\varphi_3\}$ forms a ring wrt the operations $+$ and $\cdot$ as defined in theorem (2.8). For arbitrary $x\in R$

(a) $x(\bar{o}+\bar{o})=x\bar{o}+x\bar{o}=o+o=x\bar{o}$

$x(\bar{o}\circ\bar{o})=x\bar{o}\circ x\bar{o}=o\circ o=x\bar{o}$

(b) $x(\bar{o}+\varphi_3)=x\bar{o}+x\varphi_3=o+x\varphi_3=x\varphi_3$

$x(\varphi_3+\bar{o})=x\varphi_3+x\bar{o}=x\varphi_3+o=x\varphi_3$

$x(\bar{o}\circ\varphi_3)=x\bar{o}\circ x\varphi_3=o\circ x\varphi_3=x\bar{o}$

$x(\varphi_3\circ\bar{o})=x\varphi_3\circ x\bar{o}=x\varphi_3\circ o=x\bar{o}$

(c) $x(\varphi_3+\varphi_3)=x\varphi_3+x\varphi_3=o=x\bar{o}$

$x(\varphi_3\circ\varphi_3)=x\varphi_3\circ x\varphi_3=x\varphi_3$

Hence,

$\bar{R}$:

$\bar{+}$	$\bar{o}$	φ_3
$\bar{o}$	$\bar{o}$	φ_3
φ_3	φ_3	$\bar{o}$

$\bar{\circ}$	$\bar{o}$	φ_3
$\bar{o}$	$\bar{o}$	$\bar{o}$
φ_3	$\bar{o}$	φ_3

is a ring with identity.

Clearly, $\tau_2\subset R_2x\bar{R}$, $\tau_3\subset R_3x\bar{R}$ and $\tau_4\subset R_4x\bar{R}$ defined as $o\tau_i=\bar{o}$ for $i=2,3,4$, $1\tau_2=\varphi_3$, $a\tau_3=\varphi_3$, and $(a+1)\tau_4=\varphi_4$ are isomorphisms. Define $\alpha_2=\varphi_2\tau_2, \alpha_3=\varphi_3\tau_3$ and $\alpha_4=\varphi_4\tau_3$ where for $i=2,3,4$ $x\alpha_i=(x\varphi_i)\tau_i$. Clearly, α_2,α_3 and α_4 are homomorphisms of R onto $\bar{R}$. Hence if for $x,y\in R$ we define $\#^2,\#^3$ and $\#^4$ such that $x\#^iy=x(y\alpha_i)$ for $i=2,3,4$ then $(R,+,\circ,\#^i)$ for $i=2,3,4$ are SErings with the following tables for the operations $\#^i$. (The tables were filled in by considering specific values for x,y and i in $x\#^iy=x(y\alpha_i)=x((y\varphi_i)\tau_i))$.

$\#^i$	o	1	a	a+1
o	o	o	o	o
1	o	a	o	a
a	o	a	o	a
a+1	o	o	o	o

i=2 and 4

$\#^3$	o	1	a	a+1
o	o	o	o	o
1	o	a	a	o
a	o	a	a	o
a+1	o	o	o	o

Since the tables for i=2 and i=4 are the same we have only two distinct SErings. From the tables we see that $(R,+,\cdot,\#_2)$ is a noncommutative and $(R,+,\cdot,\#_3)$ is commutative. By trying every combination of three elements $(R,+,\cdot,\#_3)$ can be shown to be associative while $(R,+,\cdot,\#_2)$ is not since $a\#_2[1\#_2(a+1)]=a\#_2a=o$ and $(a\#_2 1)\#_2(a+1)=a\#_2(a+1)=a$. Hence, $(R,+,\cdot,\#_2)$ is a noncommutative SEring and $(R,+,\cdot,\#_3)$ is a commutative Ering (ER).

(6.3) Let R be the following ring

+	o	1	a	b	a+1	b+1	a+b	a+b+1
o	o	1	a	b	a+1	b+1	a+b	a+b+1
1	1	o	a+1	b+1	a	b	a+b+1	a+b
a	a	a+1	o	a+b	1	a+b+1	b	b+1
b	b	b+1	a+b	o	a+b+1	1	a	a+1
a+1	a+1	a	1	a+b+1	o	a+b	b+1	b
b+1	b+1	b	a+b+1	1	a+b	o	a+1	a
a+b	a+b	a+b+1	b	a	b+1	a+1	o	1
a+b+1	a+b+1	a+b	b+1	a+1	b	a	1	o

$\cdot$	o	1	a	b	a+1	b+1	a+b	a+b+1
o	o	o	o	o	o	o	o	o
1	o	1	a	b	a+1	b+1	a+b	a+b+1
a	o	a	a	o	o	a	a	o
b	o	b	o	b	b	o	b	o
a+1	o	a+1	o	b	a+1	a+b+1	b	a+b+1
b+1	o	b+1	a	o	a+b+1	b+1	a	a+b+1
a+b	o	a+b	a	b	b	a	a+b	o
a+b+1	o	a+b+1	o	o	a+b+1	a+b+1	o	a+b+1

Define $\varphi_1=\bar{o}\subset Rx\{o\}$ such that for $x\in R$ $x\bar{o}=o$. As shown before $\bar{o}\subset E(R)$. Define $\varphi_2\subset Rx\{o,1,a,a+1\}$ such that $o\varphi_2=o$, $1\varphi_2=1$, $a\varphi_2=a$, $(a+1)\varphi_2=a+1$, $b\varphi_2=o$, $(b+1)\varphi_2=1$, $(a+b)\varphi_2=a$, and $(a+b+1)\varphi_2=a+1$. φ_2 is an endomorphism since if $x,y\in R$ then $(x+x)\varphi_2=o\varphi_2=o=x\varphi_2+x\varphi_2$, $(x\cdot x)\varphi_2=x\varphi_2=x\varphi_2 x\varphi_2$, (let φ denote φ_2)

(a) $x=o$: $(x+y)\varphi=y\varphi=o\varphi+y\varphi$, $(xy)\varphi=o\varphi=o\varphi y\varphi$

(b) $x=1,y=a$: $(1+a)\varphi=1+a=1\varphi+a\varphi$, $(1a)\varphi=a\varphi=1\varphi a\varphi$

$x=1,y=a+1$: $(1+a+1)\varphi=a\varphi=a=1\varphi+(a+1)\varphi$, $[1(a+1)]\varphi$ $=(a+1)\varphi=(a+1)=1\varphi(a+1)\varphi$

$x=1, y=b$: $(1+b)\varphi=1=1\varphi+b\varphi$, $(1b)\varphi=b\varphi=o=1\varphi b\varphi$

$x=1, y=b+1$: $(1+b+1)\varphi=b\varphi=o=1\varphi+(b+1)\varphi$, $[1(b+1)]\varphi$ $=(b+1)\varphi=1=1\varphi(b+1)\varphi$

$x=1, y=a+b$: $(1+a+b)\varphi=a+1=1\varphi+(a+b)\varphi$, $[1(a+b)]\varphi=$ $(a+b)\varphi=a=1\varphi(a+b)\varphi$

$x=1, y=a+b+1$: $(1+a+b+1)\varphi=(a+b)\varphi=a=1\varphi+(a+b+1)\varphi$, $[1(a+b+1)]\varphi=(a+b+1)\varphi=a+1=1\varphi(a+b+1)\varphi$

(c) $x=a, x=a+1$: $(a+a+1)\varphi=1\varphi=1=a\varphi(a+1)\varphi$, $[a(a+1)]\varphi=$ $o\varphi=o=a\varphi(a+1)\varphi$

$x=a, y=b$: $(a+b)\varphi=a=a\varphi+b\varphi$, $(ab)\varphi=o\varphi=o=a\varphi b\varphi$

$x=a, y=b+1$: $(a+b+1)\varphi=a+1=a\varphi+(b+1)\varphi$, $[a(b+1)]\varphi=$ $a=a\varphi(b+1)\varphi$

$x=a, y=a+b$: $(a+a+b)\varphi=b\varphi=o=a\varphi+(a+b)\varphi$, $[a(a+b)]\varphi$ $=a\varphi=a=a\varphi(a+b)\varphi$

$x=a, y=a+b+1$: $(a+a+b+1)\varphi=(b+1)\varphi=1=a\varphi+(a+b+1)\varphi$, $[a(a+b+1)]\varphi=o=a\varphi(a+b+1)\varphi$

(d) $x=b, x=a+1$: $(b+a+1)\varphi=a+1=b\varphi+(a+1)\varphi$, $[b(a+1)]\varphi=$ $b=b\varphi(a+1)\varphi$

$x=b, y=b+1$: $(b+b+1)\varphi=1\varphi=1=b\varphi+(b+1)\varphi$, $[b(b+1)]\varphi$ $=o\varphi=o=b\varphi(b+1)\varphi$

$x=b, y=a+b$: $(b+a+b)\varphi=a\varphi=a=b\varphi+(a+b)\varphi$, $[b(a+b)]\varphi$ $=b\varphi=o=b\varphi(a+b)\varphi$

$x=b, y=a+b+1$: $(b+a+b+1)\varphi=(a+1)\varphi=a+1=b\varphi+(a+b+1)\varphi$, $[b(a+b+1)]\varphi=o\varphi=o=b\varphi(a+b+1)\varphi$

(e) $x=a+1, y=b+1$: $(a+1+b+1)\varphi=(a+b)\varphi=a=(a+1)\varphi+(b+1)\varphi$, $[(a+1)(b+1)]\varphi=(a+b+1)\varphi=a+1=(a+1)\varphi(b+1)\varphi$

$x=a+1, y=a+b$: $(a+1+a+b)\varphi=(1+b)\varphi=1=(a+1)\varphi+(b+1)\varphi$, $[(a+1)(a+b)]\varphi=b\varphi=o=(a+1)\varphi(a+b)\varphi$

$x=a+1, y=a+b+1$: $(a+1+a+b+1)\varphi=b\varphi=o=(a+1)\varphi+(a+b+1)\varphi$, $[(a+1)(a+b+1)]\varphi=(a+b+1)\varphi=a+1=(a+1)\varphi(a+b+1)\varphi$

(f) $x=b+1, y=a+b$: $(b+1+a+b)\varphi=(1+a)\varphi=a+1=(b+1)\varphi+(a+b)\varphi$, $[(b+1)(a+b)]\varphi=a\varphi=a=(b+1)\varphi(a+b)\varphi$

$x=b+1, y=a+b+1$: $(b+1+a+b+1)\varphi=a\varphi=a=(b+1)\varphi+(a+b+1)\varphi$, $[(b+1)(a+b+1)]\varphi=(a+b+1)\varphi=a+1=(b+1)\varphi(a+b+1)\varphi$

(g) $x=a+b, y=a+b+1$: $(a+b+a+b+1)\varphi=1\varphi=1=(a+b)\varphi+(a+b+1)\varphi$, $[(a+b)(a+b+1)]\varphi=o\varphi=o=(a+b)\varphi(a+b+1)\varphi$.

Define $\varphi_3\subset R\times\{o,a\}$ such that $o\varphi_3=o$, $1\varphi_3=a$, $(a+1)\varphi_3=o$, $(b+1)\varphi_3=a$, $(a+b)\varphi_3=a$ and $(a+b+1)\varphi_3=o$. φ_3 is an endomorphism since if $x,y\in R$ then $(x+x)\varphi_3=o\varphi_3=o=x\varphi_3+x\varphi_3$, $(xx)\varphi_3=x\varphi_3=x\varphi_3x\varphi_3$, (let φ denote φ_3)

(a) $x=o$: $(x+y)\varphi=y\varphi=x\varphi+y\varphi$, $(xy)\varphi=o\varphi=x\varphi y\varphi$

(b) $x=1, y=a$: $(1+a)\varphi=o=1\varphi+a\varphi$, $(1a)\varphi=a\varphi=a=1\varphi a\varphi$

$x=1, y=b$: $(1+b)\varphi=a=1\varphi+b\varphi$, $(1b)\varphi=b\varphi=o=1\varphi b\varphi$

$x=1, y=a+1$: $(1+a+1)\varphi=a\varphi=a=1\varphi+(a+1)\varphi$, $[1(a+1)]\varphi=(a+1)\varphi=o=1\varphi(a+1)\varphi$

$x=1, y=b+1$: $(1+b+1)\varphi=b\varphi=o=1\varphi+(b+1)\varphi$, $[1(b+1)]\varphi=(b+1)\varphi=a=1\varphi(b+1)\varphi$

$x=1, y=a+b$: $(1+a+b)\varphi=o=1\varphi+(a+b)\varphi$, $[1(a+b)]\varphi=(a+b)\varphi=a=1\varphi(a+b)\varphi$

$x=1, y=a+b+1$: $(1+a+b+1)\varphi=(a+b)\varphi=a=1\varphi+(a+b+1)\varphi$, $[1(a+b+1)]\varphi=(a+b+1)\varphi=o=1\varphi+(a+b+1)\varphi$

(c) $x=a, y=b$: $(a+b)\varphi=a=a\varphi+b\varphi$, $(ab)\varphi=o\varphi=o=a\varphi b\varphi$

$x=a, y=a+1$: $(a+a+1)\varphi=1\varphi=a=a\varphi+(a+1)\varphi$, $[a(a+1)]\varphi$ $=o\varphi=o=a\varphi(a+1)\varphi$

$x=a, y=b+1$: $(a+b+1)\varphi=o=a\varphi(b+1)\varphi$, $[a(b+1)]\varphi=a=$ $a\varphi(b+1)\varphi$

$x=a, y=a+b$: $(a+a+b)\varphi=b\varphi=o=a\varphi+(a+b)\varphi$, $[a(a+b)]\varphi$ $=a\varphi=a=a\varphi(a+b)\varphi$

$x=a, y=a+b+1$: $(a+a+b+1)\varphi=(b+1)\varphi=a=a\varphi+(a+b+1)\varphi$, $[a(a+b+1)]\varphi=o\varphi=o=a\varphi(a+b+1)\varphi$

(d) $x=b, y=a+1$: $(b+a+1)\varphi=o\varphi=o=b\varphi+(a+1)\varphi$, $[b(a+1)]\varphi$ $=b\varphi=o=b\varphi(a+1)\varphi$

$x=b, y=b+1$: $(b+b+1)\varphi=1\varphi=a=b\varphi+(b+1)\varphi$, $[b(b+1)]\varphi$ $=o\varphi=o=b\varphi(b+1)\varphi$

$x=b, y=a+b$: $(b+a+b)\varphi=a\varphi=a=b\varphi+(a+b)\varphi$, $[b(a+b)]\varphi$ $=b\varphi=o=b\varphi(a+b)\varphi$

$x=b, y=a+b+1$: $(b+a+b+1)\varphi=(a+1)\varphi=o=b\varphi+(a+b+1)\varphi$, $[b(a+b+1)]\varphi=o=b\varphi(a+b+1)\varphi$

(e) $x=a+1, y=b+1$: $(a+1+b+1)\varphi=(a+b)\varphi=a=(a+1)\varphi+(b+1)\varphi$, $[(a+1)(b+1)]\varphi=(a+b+1)\varphi=o=(a+1)\varphi(b+1)\varphi$

$x=a+1, y=a+b$: $(a+1+a+b)\varphi=(b+1)\varphi=a=(a+1)\varphi+(a+b)\varphi$, $[(a+1)(a+b)]\varphi=b\varphi=o=(a+1)\varphi(a+b)\varphi$

$x=a+1, y=a+b+1$: $(a+1+a+b+1)\varphi=b\varphi=o=(a+1)\varphi+$ $(a+b+1)\varphi$, $[(a+1)(a+b+1)]\varphi=(a+b+1)\varphi=o=$ $(a+1)\varphi(a+b+1)\varphi$

(f) $x=b+1, y=a+b$: $(b+1+a+b)\varphi=(a+1)\varphi=o=(b+1)\varphi+(a+b)\varphi$, $[(b+1)(a+b)]\varphi=a\varphi=a=(b+1)\varphi(a+b)\varphi$

$x=b+1, y=a+b+1$: $(b+1+a+b+1)\varphi=a\varphi=a=(b+1)\varphi+$ $(a+b+1)\varphi$, $[(b+1)(a+b+1)]\varphi=(a+b+1)\varphi=o=$ $(b+1)\varphi(a+b+1)\varphi$

(g) $x=a+b, y=a+b+1$: $(a+b+a+b+1)\varphi=1\varphi=a=(a+b)\varphi+(a+b+1)\varphi$, $[(a+b)(a+b+1)]\varphi=o\varphi=o=(a+b)\varphi(a+b+1)\varphi$

Define $\varphi_4 \subset Rx\{o,a+1\}$ such that $o\varphi_4=o$, $1\varphi_4=a+1$, $a\varphi_4=o$, $b\varphi_4=o$, $(a+1)\varphi_4=a+1$, $(b+1)\varphi_4=a+1$, $(a+b)\varphi_4=o$, and $(a+b+1)\varphi_4=a+1$. φ_4 is an endomorphism since if $x,y \in R$ then $(x+x)\varphi_4=o=x\varphi_4+x\varphi_4$, $(xx)\varphi_4=x\varphi_4=x\varphi_4 x\varphi_4$, (let φ denote φ_4)

(a) $x=o$: $(x+y)\varphi=y\varphi=o\varphi+y\varphi$, $(xy)\varphi=o\varphi=o\varphi y\varphi$

(b) $x=1, y=a$: $(1+a)\varphi=a+1=1\varphi+a\varphi$, $(1a)\varphi=a\varphi=o=1\varphi a\varphi$

$x=1, y=b$: $(1+b)\varphi=a+1=1\varphi+b\varphi$, $(1b)\varphi=b\varphi=o=1\varphi b\varphi$

$x=1, y=a+1$: $(1+a+1)\varphi=a\varphi=o=1\varphi+(a+1)\varphi$, $[1(a+1)]\varphi=(a+1)\varphi=a+1=1\varphi(a+1)\varphi$

$x=1, y=b+1$: $(1+b+1)\varphi=b\varphi=o=1\varphi+(b+1)\varphi$, $[1(b+1)]\varphi=(b+1)\varphi=a+1=1\varphi(b+1)\varphi$

$x=1, y=a+b$: $(1+a+b)\varphi=a+1\alpha 1\varphi+(a+b)\varphi$, $[1(a+b)]\varphi=(a+b)\varphi=o=1\varphi(a+b)\varphi$

$x=1, y=a+b+1$: $(1+a+b+1)\varphi=(a+b)\varphi=o=1\varphi+(a+b+1)\varphi$, $[1(a+b+1)]\varphi=(a+b+1)\varphi=a+1=1\varphi(a+b+1)\varphi$

(c) $x=a, y=b$: $(a+b)\varphi=o=a\varphi+b\varphi$, $(ab)\varphi=o\varphi=oa\varphi b\varphi$

$x=a, y=a+1$: $(a+a+1)\varphi=1\varphi=a+1=a\varphi+(a+1)\varphi$, $[a(a+1)]\varphi=o\varphi=o=a\varphi(a+1)\varphi$

$x=a, y=b+1$: $(a+b+1)\varphi=a+1=a\varphi+(b+1)\varphi$, $[a(b+1)]\varphi=a\varphi=o=a\varphi(b+1)\varphi$

$x=a, y=a+b$: $(a+a+b)\varphi=b\varphi=o=a\varphi+(a+b)\varphi$, $[a(a+b)]\varphi=a\varphi=o=a\varphi(a+b)\varphi$

$x=a, y=a+b+1$: $(a+a+b+1)\varphi=(b+1)\varphi=a+1=a\varphi+(a+b+1)\varphi$, $[a(a+b+1)]\varphi=o\varphi=o=a\varphi(a+b+1)\varphi$

(d) $x=b, y=a+1$: $(b+a+1)\varphi=a+1=b\varphi+(a+1)\varphi$,

$[b(a+1)]\varphi=b\varphi=o=b\varphi(a+1)\varphi$

$x=b, y=b+1$: $(b+b+1)\varphi=1\varphi=a+1=b\varphi+(b+1)\varphi$,

$[b(b+1)]\varphi=o\varphi=o=b\varphi(b+1)\varphi$

$x=b, y=a+b$: $(b+a+b)\varphi=a\varphi=o=b\varphi+(a+b)\varphi$,

$[b(a+b)]\varphi=b\varphi=o=b\varphi(a+b)\varphi$

$x=b, y=a+b+1$: $(b+a+b+1)\varphi=(a+1)\varphi=a+1=b\varphi+(a+b+1)\varphi$,

$[b(a+b+1)]\varphi=o\varphi=o=b\varphi(a+b+1)\varphi$

(e) $x=b+1, y=b+1$: $(a+1+b+1)\varphi=(a+b)\varphi=o=(a+1)\varphi+(b+1)\varphi$,

$[(a+1)(b+1)]=(a+b+1)\varphi=a+1\varphi(b+1)\varphi$

$x=a+1, y=a+b$: $(a+1+a+b)\varphi=(b+1)\varphi=a+1=$

$(a+1)\varphi+(a+b)\varphi$, $[(a+1)(a+b)]\varphi=b\varphi=o=(a+1)\varphi(a+b)\varphi$

$x=a+1, y=a+b+1$: $(a+1+a+b+1)\varphi=ba=o=(a+1)\varphi+$

$(a+b+1)\varphi$, $[(a+1)(a+b+1)]\varphi=(a+b+1)\varphi=a+1=$

$(a+1)\varphi(a+b+1)\varphi$

(f) $x=b+1, y=a+b$: $(b+1+a+b)\varphi=(a+1)\varphi=a+1=(b+1)\varphi+$

$(a+b)\varphi$, $[(b+1)(a+b)]\varphi=a\varphi=o=(b+1)\varphi(a+b)\varphi$

$x=b+1, y=a+b+1$: $(b+1+a+b+1)\varphi=a\varphi=o=(b+1)\varphi+$

$(a+b+1)\varphi$, $[(b+1)(a+b+1)]\varphi=(a+b+1)=a+1=$

$(b+1)\varphi(a+b+1)\varphi$

(g) $x=a+b, y=a+b+1$: $(a+b+a+b+1)\varphi=1\varphi=a+1=(a+b)\varphi+$

$(a+b+1)\varphi$, $[(a+b)(a+b+1)]\varphi=o\varphi=o=(a+b)\varphi(a+b+1)\varphi$.

Let us now prove that $\bar{R}=\{\varphi_1=\bar{o},\varphi_2,\varphi_3,\varphi_4\}$ is a ring with the operations + and $\circ$ as defined in theorem (2.8). Let x be an arbitrary element in R. Then

(a) $x(\varphi_i+\varphi_i)=x\varphi_i+x\varphi_i=o=x\bar{o}$ for $i=1,2,3,4$

(b) $x(\varphi_i\varphi_i)=x\varphi_i x\varphi_i=x\varphi_i$ for $i=1,2,3,4$

(c) $x(\varphi_i\varphi_j)=x\varphi_i x\varphi_j=x\varphi_j x\varphi_i=x(\varphi_j\varphi_i)$ for $i,j=1,2,3,4$

(d) $x(\varphi_i+\varphi_j)=x\varphi_i+\varphi_j=x\varphi_j+x\varphi_i=x(\varphi_j+\varphi_i)$ for $i,j=1,2,3,4$

(e) $x(\varphi_i+\bar{o})=x\varphi_i+x\bar{o}=x\varphi_i$ for $i=1,2,3,4$

(f) $x(\varphi_i\bar{o})=x\varphi_i x\bar{o}=x\varphi_i o=o=x\bar{o}$ for $i=1,2,3,4$

(g) $o(\varphi_2+\varphi_3)=o=o\varphi_4$, $1(\varphi_2+\varphi_3)=1+a=1\varphi_4$, $a(\varphi_2+\varphi_3)=a+a$ $=o=a\varphi_4$, $b(\varphi_2+\varphi_3)=o+o=b\varphi_4$, $(a+1)(\varphi_2+\varphi_3)=a+1+o=(a+1)\varphi_4$, $(b+1)(\varphi_2+\varphi_3)=1+a=(b+1)\varphi_4$, $(a+b)(\varphi_2+\varphi_3)=a+a=o=(a+b)\varphi_4$, $(a+b+1)(\varphi_2+\varphi_3)=a+1+o=(a+b+1)\varphi_4$

(h) $o(\varphi_2+\varphi_4)=o+o=o\varphi_3$, $1(\varphi_2+\varphi_4)=1+a+1=1\varphi_3$, $a(\varphi_2+\varphi_4)$ $=a+o=a\varphi_3$, $b(\varphi_2+\varphi_4)=o+o=b\varphi_3$, $(a+1)(\varphi_2+\varphi_4)=a+1+a+1=o=$ $(a+1)\varphi_3$, $(b+1)(\varphi_2+\varphi_4)=1+a+1=a=(b+1)\varphi_3$, $(a+b)(\varphi_2+\varphi_4)=$ $a+o=(a+b)\varphi_3$, $(a+b+1)(\varphi_2+\varphi_4)=a+1+a+1=o=(a+b+1)\varphi_3$

(i) $o(\varphi_3+\varphi_4)=o+o=o\varphi_2$, $1(\varphi_3+\varphi_4)=a+a+1=1=1\varphi_2$, $a(\varphi_3+\varphi_4)=a+o=a\varphi_2$, $b(\varphi_3+\varphi_4)=o+o=b\varphi_3$, $(a+1)(\varphi_3+\varphi_4)=o+a+1=$ $(a+1)\varphi_2$, $(b+1)(\varphi_3+\varphi_4)=a+a+1=1=(b+1)\varphi_2$, $(a+b)(\varphi_3+\varphi_4)=a+o=$ $(a+b)\varphi_2$, $(a+b+1)(\varphi_3+\varphi_4)=o+a+1=(a+b+1)\varphi_2$

(j) $o(\varphi_2\varphi_3)=oo=o\varphi_3$, $1(\varphi_2\varphi_3)=1a=1\varphi_3$, $a(\varphi_2\varphi_3)=aa=a=$ $a\varphi_3$, $b(\varphi_2\varphi_3)=oo=b\varphi_3$, $(a+1)(\varphi_2\varphi_3)=(a+1)o=o=(a+1)\varphi_3$, $(b+1)(\varphi_2\varphi_3)=1a=(b+1)\varphi_3$, $(a+b)(\varphi_2\varphi_3)=aa=a=(a+b)\varphi_3$, $(a+b+1)(\varphi_2\varphi_3)=(a+1)o=o=(a+b+1)\varphi_3$

(k) $o(\varphi_2\varphi_4)=oo=o\varphi_4$, $1(\varphi_2\varphi_4)=1(a+1)=(a+1)\varphi_4$, $a(\varphi_2\varphi_4)$ $=ao=o=a\varphi_4$, $b(\varphi_2\varphi_4)=oo=b\varphi_4$, $(a+1)(\varphi_2\varphi_4)=(a+1)(a+1)=a+1=$

$(a+b)\varphi_4$, $(b+1)(\varphi_2\varphi_4)=1(a+1)=(b+1)\varphi_4$, $(a+b)(\varphi_2\varphi_4)=ao=o=$

$(a+b)\varphi_4$, $(a+b+1)(\varphi_2\varphi_4)=(a+1)(a+1)=a+1=(a+b+1)\varphi_4$

(l) $o(\varphi_3\varphi_4)=oo=o\bar{o}$, $1(\varphi_3\varphi_4)=a(a+1)=o=1\bar{o}$, $a(\varphi_3\varphi_4)=ao$ $=o=a\bar{o}$, $b(\varphi_3\varphi_4)=oo=b\bar{o}$, $(a+1)(\varphi_3\varphi_4)=o(a+1)=o=(a+1)\bar{o}$, $(b+1)(\varphi_3\varphi_4)=a(a+1)=o=(a+1)\bar{o}$, $(a+b)(\varphi_3\varphi_4)=ao=o=(a+b)\bar{o}$, $(a+b+1)(\varphi_3\varphi_4)=o(a+1)=o=(a+b+1)\bar{o}$.

Hence, $\bar{R}$ is a ring with identity and has the following tables for + and $\cdot$.

+	$\bar{o}$	φ_2	φ_3	φ_4
$\bar{o}$	$\bar{o}$	φ_2	φ_3	φ_4
φ_2	φ_2	$\bar{o}$	φ_4	φ_3
φ_3	φ_3	φ_4	$\bar{o}$	φ_2
φ_4	φ_4	φ_3	φ_2	$\bar{o}$

$\cdot$	$\bar{o}$	φ_2	φ_3	φ_4
$\bar{o}$	$\bar{o}$	$\bar{o}$	$\bar{o}$	$\bar{o}$
φ_2	$\bar{o}$	φ_2	φ_3	φ_4
φ_3	$\bar{o}$	φ_3	φ_3	$\bar{o}$
φ_4	$\bar{o}$	φ_4	$\bar{o}$	φ_4

Now $R'=\{o,1,a,a+1\}$ with its addition and multiplication tables on page 70 is a subring of R with identity. In fact φ_2 is an endomorphism of R onto R'. Define $\tau \subset R' x \bar{R}$ such that $o\tau=\bar{o}$, $1\tau=\varphi_2$, $a\tau=\varphi_3$, and $(a+1)\tau$ $=\varphi_4$. Clearly, τ is an isomorphism of R' onto $\bar{R}$. Hence, by theorem (2.12) if we define # such that for $x,y \in R$ $x\#y=x(y\alpha)=x((y\varphi_2)\tau)$ then $(R,+,\cdot,\#)$ is a SEring with the following table for # since $\alpha=\varphi_2\tau$ is a homomorphism of R onto $\bar{R}$.

+	o	1	a	a+1
o	o	1	a	a+1
1	1	o	a+1	a
a	a	a+1	o	1
a+1	a+1	a	1	o

.	o	1	a	a+1
o	o	o	o	o
1	o	1	a	a+1
a	o	a	a	o
a+1	o	a+1	o	a+1

Addition and multiplication tables for R'.

#	o	1	a	b	a+1	b+1	a+b	a+b+1
o	o	o	o	o	o	o	o	o
1	o	1	a	o	a+1	1	a	a+1
a	o	a	a	o	o	a	a	o
b	o	o	o	o	o	o	o	o
a+1	o	a+1	o	o	a+1	a+1	o	a+1
b+1	o	1	a	o	a+1	1	a	a+1
a+b	o	a	a	o	o	a	a	o
a+b+1	o	a+1	o	o	a+1	a+1	o	a+1

table for (R,+,∘,#)

Section 7. Examples of subBEBRs and of homomorphisms

(7.1) SubRER of a RER. Let R be the RER in example (5.2). Let $S=F\subset R$. If $x,y\in S$ then $x-y\in S$, $x\circ y\in S$, and $x\#y\in S$. Hence, by theorem (4.3) S is a subRER.

(7.2) SubSER of a SER. In example (6.3) let $S=\{o,1,b,b+1\}\subset R$. Then the following tables show that S is a subSER of R.

+	o	1	b	b+1
o	o	1	b	b+1
1	1	o	b+1	b
b	b	b+1	o	1
b+1	b+1	b	1	o

∘	o	1	b	b+1
o	o	o	o	o
1	o	1	b	b+1
b	o	b	b	o
b+1	o	b+1	o	b+1

#	o	1	b	b+1
o	o	o	o	o
1	o	1	o	1
b	o	o	o	o
b+1	o	1	o	1

(7.3) Homomorphism of a RER into a RER. In example (7.1) define $\varphi \subset R \times S$ such that for

$$a = \sum_{i=o}^{n_a} a_i x^i \in R$$

$a\varphi = a_o$. As shown in example (5.9) in chapter II φ is a ring homomorphism. If $a, b \in R$ then $(a\#b)\varphi = (a\varphi)\#(b\varphi)$. Hence, φ is a homomorphism of R into S. Note that φ is also an M-homomorphism as shown in example (5.9) in chapter II.

(7.4) Homomorphism of a SER into a SER. Let S be the subSER of the SER R in example (7.2). Then by tedious work it can be shown that $\beta \subset R \times S$ such that $o\beta = o$, $1\beta = 1$, $b\beta = b$, $(b+1)\beta = b+1$, $a\beta = o$, $(a+1)\beta = 1$, $(a+b)\beta = b$, and $(a+b+1)\beta = b+1$ is a homomorphism (in fact β is an endomorphism). Note that in this case β is not an M-homomorphism since $[(b+1)\#(a+1)]\beta = (a+1)\beta = 1$ where as $(b+1)\beta\#(a+1) = (b+1)\#(a+1) = a+1 \notin S$.

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